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Spectra of PP-wave limits of M-/Superstring theory on $\mathrm{AdS}_p \times \mathrm{S}^q$ spaces*

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Abstract: In this paper we show how one can obtain very simply the spectra of the PP-wave limits of M-theory over $\mathrm{AdS}_{7(4)} \times \mathrm{S}^{4(7)}$ spaces and IIB superstring theory over $\mathrm{AdS}_5 \times \mathrm{S}^5$ from the oscillator construction of the Kaluza-Klein spectra of these theories over the corresponding spaces. The PP-wave symmetry superalgebras are obtained by taking the number P of "colors" of oscillators to be large (infinite). In this large P limit, the symmetry superalgebra $\mathfrak{osp}(8^*|4)$ of $\mathrm{AdS}_7 \times \mathrm{S}^4$ and the symmetry superalgebra $\mathfrak{osp}(8|4,\mathbb{R})$ of $\mathrm{AdS}_4 \times \mathrm{S}^7$ lead to isomorphic PP-wave algebras, which is $\mathfrak{su}(4|2) \circledS \mathfrak{H}^{18,16}$, while the symmetry superalgebra $\mathfrak{su}(2,2|4)$ of $\mathrm{AdS}_5 \times \mathrm{S}^5$ leads to $[\mathfrak{psu}(2|2) \oplus \mathfrak{psu}(2|2) \oplus \mathfrak{u}(1)] \circledS \mathfrak{H}^{16,16}$ as its PP-wave algebra [$\mathfrak{H}^{m,n}$ denoting a super-Heisenberg algebra with m bosonic and n fermionic generators]. The zero mode spectra of M-theory or IIB superstring theory in the PP-wave limit correspond simply to the unitary positive energy representations of these algebras whose lowest weight vector is the Fock vacuum of all the oscillators. General positive energy supermultiplets including those corresponding to higher modes can similarly be constructed by the oscillator method.

Keywords: Superstrings and Heterotic Strings, M-Theory, Penrose limit and pp-wave background, AdS-CFT and dS-CFT Correspondence.

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Contents

1. Introduction **1**

2. Review of the oscillator construction of the positive energy unitary supermultiplets of AdS/Conformal superalgebras **4**

2.1 Oscillator construction of the positive energy representations of $SU(2,2|4)$ 6

2.1.1 $SU(2,2)$ representations via the oscillator method 6

2.1.2 $SU(4)$ representations via the oscillator method 8

2.1.3 Unitary representations of $SU(2,2|4)$ via the oscillator method 8

2.2 Oscillator construction of the positive energy representations of $OSp(8^*|4)$ 12

2.2.1 $SO^*(8) \approx SO(6,2)$ representations via the oscillator method 12

2.2.2 $USp(4) \approx SO(5)$ representations via the oscillator method 13

2.2.3 Unitary representations of $OSp(8^*|4)$ via the oscillator method 14

2.3 Oscillator construction of the positive energy representations of $OSp(8|4,\mathbb{R})$ 16

2.3.1 $SO(3,2) \approx Sp(4,\mathbb{R})$ representations via the oscillator method 16

2.3.2 $SO(8)$ representations via the oscillator method 17

2.3.3 Unitary representations of $OSp(8|4,\mathbb{R})$ via the oscillator method 18

3. Contraction of the superalgebras over $AdS_p \times S^q$ spaces to maximally supersymmetric PP-wave algebras in the oscillator formalism **20**

4. Contraction of $\mathfrak{su}(2,2|4)$ and the PP-wave spectrum in the oscillator formalism **22**

5. Contraction of M-theory superalgebras $\mathfrak{osp}(8^*|4)$ and $\mathfrak{osp}(8|4,\mathbb{R})$ and the PP-wave spectrum in the oscillator formalism **25**

A. Light-cone quantization of IIB superstring and the oscillator formalism **27**

1. Introduction

It has recently been shown that the type IIB superstring theory on the PP-wave background is exactly solvable [1, 2]. This PP-wave background geometry can be obtained by taking a particular Penrose limit of $AdS_5 \times S^5$ [3]–[7]. The PP-wave metric has the simple form

$$ds^2 = 2dx^+dx^- - \frac{\mu^2}{2}\left(\sum_{I=1}^{8} {x_I}^2\right)\left(dx^+\right)^2 + \sum_{I=1}^{8} {dx_I}^2 \,, \tag{1.1}$$

JHEP10(2002)007

where $x^{\pm} = \frac{1}{\sqrt{2}}\left(x^9 \pm x^0\right)$ and μ is related to the R-R 5-form field strength as

$$F_{+1234} = F_{+5678} = \frac{\mu}{4\pi^3 g_s \alpha'^2}\,. \tag{1.2}$$

The string coupling constant g_s is given by the exponential of the dilaton field $g_s = e^{\Phi}$. The non-vanishing 5-form field strength breaks the manifest SO(8) symmetry of the PP-wave metric down to its $\mathrm{SO}(4)\times\mathrm{SO}(4)'$ subgroup. Berenstein, Maldacena and Nastase [8] showed that the states of the IIB superstring theory in this PP-wave background geometry can be mapped to a certain sector of $\mathcal{N} = 4$ SU(N) super Yang-Mills theory corresponding to a particular limit of the usual $\mathrm{AdS}_5/\mathrm{CFT}_4$ duality. This sector of $\mathcal{N} = 4$ super Yang-Mills theory contains operators with large R-charge J (with respect to a certain U(1) subgroup of the R-symmetry group $\mathrm{SU}(4) \approx \mathrm{SO}(6)$) in the limit $N,\ J \to \infty$ with $J \sim \sqrt{N}$, such that $(E - J)$ is finite (E being the conformal dimension of the operator). More precisely, one has

$$E - J = \frac{2p^-}{\mu}\,, \qquad J = \mu p^+ R^2\,, \qquad R^4 = 4\pi\alpha'^2 g_s N\,, \tag{1.3}$$

where $p^{\pm}$ are the light-cone momenta ($-p^-$ is the light-cone energy). Following the work of BMN [8], a large number of papers on PP-wave limits of M-/superstring theory appeared in the literature [9]–[64].

The IIB superstring theory in PP-wave background was quantized in [2], where in particular the spectrum of states of the zero mode sector was obtained and lifted into representations of $\mathrm{SO}(4) \times \mathrm{SO}(4)'$. The states created by these zero mode operators of the PP-wave limit are the analogs of the massless supergravity modes of IIB superstring in 10-dimensional Minkowski spacetime background.

The spectrum of IIB supergravity over $\mathrm{AdS}_5 \times \mathrm{S}^5$ (with the symmetry supergroup $\mathrm{SU}(2,2|4)$) was obtained in [65, 66]. In [65] the entire Kaluza-Klein tower was obtained by a simple tensoring of CPT self-conjugate doubleton supermultiplet of the AdS supergroup $\mathrm{SU}(2,2|4)$ with itself repeatedly. This CPT self-conjugate doubleton supermultiplet of $\mathrm{SU}(2,2|4)$ does not have a Poincaré limit in AdS_5 and is simply the $\mathcal{N} = 4$ super Yang-Mills multiplet in $D = 4$ Minkowski space which can be identified with the boundary of AdS_5. The tower of Kaluza-Klein states listed in table 1 of [65] was given in the compact subsupergroup basis $\mathrm{SU}(2|2) \times \mathrm{SU}(2|2) \times \mathrm{U}(1)$ of $\mathrm{SU}(2,2|4)$, with respect to which the Lie superalgebra $\mathfrak{g}$ of $\mathrm{SU}(2,2|4)$ has a 3-graded decomposition

$$\mathfrak{g} = \mathfrak{g}^{(-1)} + \mathfrak{g}^{(0)} + \mathfrak{g}^{(+1)}\,, \tag{1.4}$$

where $\mathfrak{g}^{(0)} = \mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2) \oplus \mathfrak{u}(1) = \mathfrak{psu}(2|2) \oplus \mathfrak{psu}(2|2) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)$.[1]

In this paper, we first study the PP-wave limits of the oscillator construction of the positive energy unitary supermultiplets of $\mathrm{SU}(2,2|4)$. In particular, we show how one can obtain very simply the zero mode spectrum of IIB superstrings in PP-wave geometry from the Kaluza-Klein spectrum of IIB supergravity over $\mathrm{AdS}_5 \times \mathrm{S}^5$. The PP-wave algebra is

[1] A linear combination of the U(1) generators commutes with all the other generators of $\mathrm{SU}(2,2|4)$ and hence acts as a central charge. The superalgebra obtained by modding out this U(1) is denoted as $\mathrm{PSU}(2,2|4)$. The tower of K-K supermultiplets of IIB supergravity carries zero central charge.

obtained by taking the number P of colors of super-oscillators to be very large ($P \to \infty$). In the PP-wave limit, the contraction of $\mathfrak{psu}(2,2|4)$ is the semi-direct sum of the compact subsuperalgebra $\mathfrak{psu}(2|2) \oplus \mathfrak{psu}(2|2) \oplus \mathfrak{u}(1)$ with a super-Heisenberg algebra $\mathfrak{H}^{16,16}$ which consists of 16 bosonic and 16 fermionic generators (plus the central charge). The entire zero mode spectrum of IIB superstring theory in the PP wave background corresponds to the unitary supermultiplet of this contracted algebra whose lowest weight vector is simply the Fock vacuum of all the super-oscillators, in agreement with the results of Metsaev and Tseytlin [1, 2].

The range of eigenvalues $\mathcal{E}_0$ of the PP-wave hamiltonian H on the fields of the zero mode PP-wave supermultiplet is $\Delta\mathcal{E}_0 = 8$ in our units and normalization.[2] On the other hand, the range of eigenvalues of the operator that becomes the PP-wave hamiltonian in the limit can be less than eight for low-lying ($P < 4$) Kaluza-Klein supermultiplets or for doubleton supermultiplets of $\mathrm{SU}(2,2|4)$. For example, the massless graviton supermultiplet of $\mathrm{SU}(2,2|4)$ has $\Delta\mathcal{E}_0 = 4$ and the doubleton supermultiplet has $\Delta\mathcal{E}_0 = 2$. There are no analogs of supermultiplets of $\mathrm{SU}(2,2|4)$ with $\Delta\mathcal{E}_0 < 8$ in the PP wave limit.

Next we study the PP-wave limit of the oscillator construction of the M-theory superalgebras over $\mathrm{AdS}_7 \times \mathrm{S}^4$ and $\mathrm{AdS}_4 \times \mathrm{S}^7$. The Kaluza-Klein spectra of 11-dimensional supergravity over $\mathrm{AdS}_7 \times \mathrm{S}^4$ and $\mathrm{AdS}_4 \times \mathrm{S}^7$ were fitted into unitary supermultiplets of $\mathrm{OSp}(8^*|4)$ and $\mathrm{OSp}(8|4,\mathbb{R})$ a long time ago in [69, 70], where the oscillator construction of the corresponding unitary supermultiplets was given with respect to their maximal compact subsuperalgebras $\mathfrak{u}(4|2)$ and $\mathfrak{u}(2|4)$, respectively. The PP-wave contraction of the oscillator realization of $\mathrm{OSp}(8^*|4)$ and $\mathrm{OSp}(8|4,\mathbb{R})$ require again taking the number P of colors of oscillators to be large ($P \to \infty$). Their PP-wave limits result in isomorphic superalgebras, which is the semi-direct sum of $\mathfrak{su}(4|2)$ (or $\mathfrak{su}(2|4)$) with a super-Heisenberg algebra $\mathfrak{H}^{18,16}$ with 18 bosonic and 16 fermionic generators. Again, the spectrum of the zero mode sector of M-theory PP-wave algebra can be easily obtained from the Kaluza-Klein spectra given in [69, 70]. The range of eigenvalues of the hamiltonian of the zero mode supermultiplet of the M-theory PP-wave algebra is also $\Delta\mathcal{E}_0 = 8$ and once again, there do not exist any analogs of $\mathrm{OSp}(8^*|4)$ and $\mathrm{OSp}(8|4,\mathbb{R})$ supermultiplets with the range of eigenvalues $\Delta\mathcal{E}_0 < 8$ in the PP-wave limit.

The plan of the paper is as follows. In section 2, we review the oscillator construction of the positive energy unitary supermultiplets of the relevant AdS/Conformal superalgebras, i.e. those of $\mathfrak{su}(2,2|4)$, $\mathfrak{osp}(8^*|4)$ and $\mathfrak{osp}(8|4,\mathbb{R})$. We also give the Kaluza-Klein spectra of IIB supergravity on S^5 and of 11-dimensional supergravity on S^4 and S^7 following [65, 69, 70].

In section 3, we show how to take the PP-wave limits of these superalgebras as $P \to \infty$, while preserving all the supersymmetries. Remarkably, this contraction preserving maximal supersymmetry also determines the parameter $\rho = R_{\mathrm{AdS}}/R_{\mathrm{S}}$ in agreement with the results of [7]. This parameter ρ is determined by the linear combination of the two $\mathrm{U}(1)$ generators (E and J) that is independent of the number P of colors.

[2] Here we are considering the eigenvalues of the hamiltonian H on the lowest weight vector of the positive energy unitary representation corresponding to each field of the supermultiplet. Clearly the higher Fourier modes of these fields can have arbitrarily large energies.

JHEP10(2002)007

In section 4, we give the contraction of $\mathfrak{su}(2,2|4)$ to the PP-wave algebra by rescaling the generators of $\mathfrak{g}^{(-1)}$ and $\mathfrak{g}^{(+1)}$ subspaces and the P-dependent linear combination of E and J and then by taking the limit $P \to \infty$. In this limit, the above $\mathfrak{g}^{(\pm 1)}$ subspace generators become the generators of a super-Heisenberg algebra and the P-dependent linear combination of E and J, into which $\mathfrak{g}^{(-1)}$ and $\mathfrak{g}^{(+1)}$ generators close under super commutation becomes a central charge. The unitary supermultiplet of the PP-wave algebra determined by taking the vacuum of all bosonic and fermionic oscillators as the lowest weight vector is the zero mode supermultiplet of IIB superstring theory in PP-wave background. The construction of the general unitary supermultiplets of the PP-wave algebra proceeds as in the case of $\mathfrak{su}(2,2|4)$.

In section 5, we give the PP-wave contraction of $\mathfrak{osp}(8^*|4)$ and $\mathfrak{osp}(8|4,\mathbb{R})$ by a similar rescaling and then taking the same limit $P \to \infty$. We then give the unitary supermultiplet of the M-theory PP-wave algebra defined by taking the vacuum of all bosonic and fermionic oscillators as the lowest weight vector. This supermultiplet is the zero mode supermultiplet of maximally supersymmetric M-theory PP-wave algebra. General unitary supermultiplets of M-theory superalgebra are obtained by choosing general representations of the maximal compact subsupergroup as the lowest representation.

Appendix A gives the dictionary between our oscillators and those of Metsaev and Tseytlin [2] for IIB theory.

2. Review of the oscillator construction of the positive energy unitary supermultiplets of AdS/Conformal superalgebras

The general oscillator method for constructing the unitary irreducible representations (UIRs) of the lowest (or highest) weight type of non-compact groups was given in [67]. This method yields the lowest weight, positive energy UIRs of a non-compact group (belonging to the holomorphic discrete series) over the Fock space of a set of bosonic oscillators. One realizes the generators of the non-compact group as bilinears of these bosonic oscillators that transform in a certain finite dimensional representation of its maximal compact subgroup. The minimal realization of these generators requires either one or two sets (depending on the non-compact group) of bosonic annihilation and creation operators transforming irreducibly under its maximal compact subgroup. These minimal positive energy representations are fundamental in the sense that all the other positive energy UIRs belonging to the holomorphic discrete series can be obtained from these minimal representations by a simple tensoring procedure. These fundamental UIRs are called singletons or doubletons, respectively, depending on whether the minimal realization requires one or two sets of such oscillators [65, 69, 70].

The general oscillator construction of the lowest (or highest) weight representations of non-compact supergroups was given in [68]. It was further developed and applied to the calculation of spectra of Kaluza-Klein supergravity theories in [65, 69, 70] and to AdS/CFT dualities in [71]–[75].

A simple non-compact group G that admits unitary representations of the lowest weight type has a maximal compact subgroup $G^{(0)}$, such that $G/G^{(0)}$ is a hermitian symmetric space. This compact subgroup $G^{(0)}$ has an abelian factor, i.e. $G^{(0)} = H \times \mathrm{U}(1)$. The Lie algebra $\mathfrak{g}$ of G has a 3-grading with respect to the Lie algebra $\mathfrak{g}^{(0)}$ of $G^{(0)}$:

$$\mathfrak{g} = \mathfrak{g}^{(-1)} \oplus \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(+1)} , \tag{2.1}$$

which simply means that the commutators of elements of grade k and l $(= 0, \pm 1)$ satisfy

$$\left[\mathfrak{g}^{(k)}, \mathfrak{g}^{(l)}\right] \subseteq \mathfrak{g}^{(k+l)} \tag{2.2}$$

with $\mathfrak{g}^{(k+l)} = 0$ for $|k + l| > 1$.

The 3-grading is determined by the generator E of the U(1) factor of the maximal compact subgroup:

$$\begin{aligned} \left[E, \mathfrak{g}^{(+1)}\right] &= \mathfrak{g}^{(+1)} \\ \left[E, \mathfrak{g}^{(-1)}\right] &= -\mathfrak{g}^{(-1)} \\ \left[E, \mathfrak{g}^{(0)}\right] &= 0 . \end{aligned} \tag{2.3}$$

If E is the energy operator, then the lowest weight UIRs correspond to positive energy representations. To construct these representations in the Fock space $\mathcal{H}$ of all the oscillators, one chooses a set of states $|\Omega\rangle$ which transform irreducibly under $H \times \mathrm{U}(1)$ and is annihilated by all the generators in $\mathfrak{g}^{(-1)}$ subspace. Then by acting on $|\Omega\rangle$ with the generators in $\mathfrak{g}^{(+1)}$, one obtains an infinite set of states

$$|\Omega\rangle , \qquad \mathfrak{g}^{(+1)} |\Omega\rangle , \qquad \mathfrak{g}^{(+1)}\mathfrak{g}^{(+1)} |\Omega\rangle , \ldots \tag{2.4}$$

which forms a UIR of the lowest weight (positive energy) type of G. Any two $|\Omega\rangle$ that transform in the same irreducible representation of $H \times \mathrm{U}(1)$ will lead to an equivalent UIR of G.

The irreducibility of the resulting representation of the non-compact group G follows from the irreducibility of the "lowest representation" $|\Omega\rangle$ with respect to the maximal compact subgroup $G^{(0)}$.[3]

The non-compact supergroups similarly admit either singleton or doubleton supermultiplets corresponding to some minimal fundamental UIRs, in terms of which one can construct all the other UIRs of the lowest weight type, belonging to the holomorphic discrete series, by a simple tensoring procedure. For example, the non-compact supergroup $\mathrm{OSp}(2N|2M, \mathbb{R})$, with the even subsupergroup $\mathrm{SO}(2N) \times \mathrm{Sp}(2M, \mathbb{R})$, admits singleton supermultiplets, while $\mathrm{OSp}(2N^*|2M)$ and $\mathrm{SU}(N, M|P)$, with even subsupergroups $\mathrm{SO}^*(2N) \times \mathrm{USp}(2M)$ and $\mathrm{SU}(N, M) \times \mathrm{SU}(P) \times \mathrm{U}(1)$ respectively, admit doubleton supermultiplets [65, 69, 70].

[3]We should note that, in the earlier literature we sometimes referred to $|\Omega\rangle$ as "lowest weight vector". However, we stress that it in general consists of a set of states in the Fock space that transform irreducibly under $G^{(0)}$.

2.1 Oscillator construction of the positive energy representations of SU(2, 2|4)

SU(2, 2|4), with the even subgroup SU(2, 2) × SU(4) × U(1) is the symmetry group of type IIB superstring theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$. The construction of UIRs of this supergroup has been studied extensively in literature using the oscillator method [65, 71, 72] as well as other methods [76, 77, 78]. What follows is only a brief summary of the oscillator construction and we refer the reader to the above references for a complete account of the method.

2.1.1 SU(2, 2) representations via the oscillator method

Unitary representations of the covering group SU(2, 2) of the conformal group SO(4, 2) in $d = 4$ have been studied extensively [79]. The group SO(4, 2) is also the AdS group of $d = 5$ spacetime with lorentzian signature. In this section we shall review the oscillator construction of the positive energy unitary representations of SU(2, 2) belonging to the holomorphic discrete series [68, 65, 71, 72].

We denote the two SU(2) subgroups of SU(2, 2) as $\mathrm{SU}(2)_L$ and $\mathrm{SU}(2)_R$ respectively. The generator E of the abelian factor in the maximal compact subgroup of SU(2, 2) is the AdS energy operator in $d = 5$ or the conformal hamiltonian in $d = 4$. To construct the relevant positive energy representations, we realize the generators of SU(2, 2) as bilinears of an arbitrary number P ("generations" or "colors") pairs of bosonic oscillators transforming in the fundamental representation of the two SU(2) subgroups. They satisfy the canonical commutation relations

$$\left[a_i(K), a^j(L)\right] = \delta_i^j \delta_{KL} , \qquad [b_r(K), b^s(L)] = \delta_r^s \delta_{KL} , \tag{2.5}$$

where $i, j = 1, 2$ and $r, s = 1, 2$ while all the other commutators among them vanish. $K, L = 1, \ldots, P$ denote the color index.

The above bosonic oscillators with an upper index (e.g. $a^i(K)$ or $b^r(K)$) denote creation operators and those with a lower index (e.g. $a_i(K)$ or $b_r(K)$) denote annihilation operators. The vacuum vector is annihilated by all the annihilation operators:

$$a_i(K) \left|0\right\rangle = 0 = b_r(K) \left|0\right\rangle \tag{2.6}$$

for all values of i, r, K.

The non-compact generators of SU(2, 2) are then realized as the following bilinears:

$$A_{ir} = \vec{a}_i \cdot \vec{b}_r , \qquad A^{ir} = \vec{a}^i \cdot \vec{b}^r \tag{2.7}$$

where

$$\vec{a}_i \cdot \vec{b}_r = \sum_{K=1}^{P} a_i(K) b_r(K) , \qquad \text{etc.} \tag{2.8}$$

They close into the generators of the compact subgroup $SU(2)_L \times SU(2)_R \times U(1)$:

$$\begin{aligned}
\left[A_{ir}, A^{js}\right] &= \delta_r^s L^j{}_i + \delta_i^j R^s{}_r + \delta_i^j \delta_r^s E \\
\left[A_{ir}, A_{js}\right] &= 0 = \left[A^{ir}, A^{js}\right] ,
\end{aligned} \tag{2.9}$$

where

$$\begin{aligned}
L^i{}_j &= \vec{a}^i \cdot \vec{a}_j - \frac{1}{2}\delta^i_j \vec{a}^l \cdot \vec{a}_l \\
R^r{}_s &= \vec{b}^r \cdot \vec{b}_s - \frac{1}{2}\delta^r_s \vec{b}^t \cdot \vec{b}_t \\
E &= \frac{1}{2}\left(\vec{a}^i \cdot \vec{a}_i + \vec{b}_r \cdot \vec{b}^r\right) .
\end{aligned} \tag{2.10}$$

$L^i{}_j$ and $R^r{}_s$ above are the generators of $SU(2)_L$ and $SU(2)_R$, respectively.

The generator E of $U(1)$ (energy operator) can be written as

$$E = \frac{1}{2}\left(N_a + N_b\right) + P \tag{2.11}$$

where $N_a = \vec{a}^i \cdot \vec{a}_i$ and $N_b = \vec{b}^r \cdot \vec{b}_r$ are the number operators corresponding to a- and b-type oscillators.

As stated before, the positive energy UIRs of $SU(2,2)$ are uniquely defined by the lowest representations $|\Omega\rangle$ that transform irreducibly under the maximal compact subgroup $S(U(2) \times U(2))$ and are annihilated by all the generators of $\mathfrak{g}^{(-1)}$ (i.e. by all A_{ir}):

$$A_{ir}\,|\Omega\rangle = 0 \qquad \text{for all } i, r. \tag{2.12}$$

Then by acting on $|\Omega\rangle$ repeatedly with the generators of $\mathfrak{g}^{(+1)}$ (i.e. with A^{ir}) one generates an infinite set of states

$$|\Omega\rangle\ , \qquad A^{ir}\,|\Omega\rangle\ , \qquad A^{ir}A^{js}\,|\Omega\rangle\ , \ldots \tag{2.13}$$

that forms the basis of the corresponding UIR of $SU(2,2)$ [68, 67, 65]. These UIRs can be identified with fields in AdS_5 or conformal fields in $d = 4$ [65, 71, 72].

The minimal oscillator realization of $SU(2,2)$ requires a pair of oscillators, i.e. $P = 1$. The resulting representations are the doubleton representations and they do not have a Poincaré limit in $d = 5$ [65, 71, 72]. The possible lowest weight vectors in this case are of the form

$$a^{i_1} \ldots a^{i_{n_L}}\,|0\rangle \qquad \text{and} \quad b^{r_1} \ldots b^{r_{n_R}}\,|0\rangle\ , \tag{2.14}$$

where n_L and n_R are some non-negative integers. The Poincaré mass operator in $d = 4$ vanishes identically for these representations and hence doubletons are massless conformal fields in four dimensions [72].

The massless representations of the AdS_5 group $SU(2,2)$ are obtained by taking two pairs ($P = 2$) of oscillators. For $P > 2$, the resulting representations of $SU(2,2)$, considered as the AdS_5 group, are all massive. Considered as the four dimensional conformal group, all the UIRs of $SU(2,2)$ with $P \geq 2$ correspond to massive conformal fields [65, 71, 72].

2.1.2 SU(4) representations via the oscillator method

The Lie algebra $\mathfrak{su}(4)$ has a 3-graded decomposition with respect to its subalgebra $\mathfrak{g}^{(0)} = \mathfrak{su}(2)\oplus\mathfrak{su}(2)\oplus\mathfrak{u}(1)$. The SU(4) generators can be realized as bilinears of P pairs of fermionic oscillators α and β, that transform in the fundamental representations of the two SU(2), which we denote as $SU(2)_{k_1}$ and $SU(2)_{k_2}$, respectively. These fermionic oscillators satisfy the canonical anticommutation relations

$$\left\{\alpha_\gamma(K), \alpha^\delta(L)\right\} = \delta^\delta_\gamma \delta_{KL}\,, \qquad \{\beta_\mu(K), \beta^\nu(L)\} = \delta^\nu_\mu \delta_{KL}\,, \tag{2.15}$$

where $\gamma, \delta = 1, 2$ and $\mu, \nu = 1, 2$ while all the other anticommutators vanish. Then the SU(4) generators are realized as:

$$\begin{aligned}
A_{\gamma\mu} &= \vec{\alpha}_\gamma \cdot \vec{\beta}_\mu\,, \qquad\qquad A^{\gamma\mu} = \vec{\alpha}^\gamma \cdot \vec{\beta}^\mu \\
M^\gamma{}_\delta &= \vec{\alpha}^\gamma \cdot \vec{\alpha}_\delta - \frac{1}{2}\delta^\gamma_\delta N_\alpha \\
S^\mu{}_\nu &= \vec{\beta}^\mu \cdot \vec{\beta}_\nu - \frac{1}{2}\delta^\mu_\nu N_\beta \\
C &= \frac{1}{2}(N_\alpha + N_\beta) - P\,,
\end{aligned} \tag{2.16}$$

where $N_\alpha = \vec{\alpha}^\gamma \cdot \vec{\alpha}_\gamma$ and $N_\beta = \vec{\beta}^\mu \cdot \vec{\beta}_\mu$ are the fermionic number operators. The bilinear operators $A_{\gamma\mu}$ and $A^{\gamma\mu}$ belong to the subspaces $\mathfrak{g}^{(-1)}$ and $\mathfrak{g}^{(+1)}$, respectively, and hence satisfy the following commutation relations:

$$\begin{aligned}
\left[A_{\gamma\mu}, A^{\delta\nu}\right] &= \delta^\nu_\mu M^\delta{}_\gamma + \delta^\delta_\gamma S^\nu{}_\mu + \delta^\nu_\mu \delta^\delta_\gamma C \\
[A_{\gamma\mu}, A_{\delta\nu}] = 0\,, &\qquad \left[A^{\gamma\mu}, A^{\delta\nu}\right] = 0\,.
\end{aligned} \tag{2.17}$$

One can construct the representations of SU(4) in the SU(2) × SU(2) × U(1) basis by choosing a set of states $|\Omega\rangle$ in the Fock space of the fermions that transforms irreducibly under $\mathfrak{g}^{(0)}$ and is annihilated by all the $\mathfrak{g}^{(-1)}$ generators; $A_{\gamma\mu}|\Omega\rangle = 0$. Then by acting on $|\Omega\rangle$ with $\mathfrak{g}^{(+1)}$ generators $A^{\mu\gamma}$ repeatedly, one creates a *finite* number of states (because of the fermionic nature of the oscillators) that form a basis of an irreducible representation of SU(4).

2.1.3 Unitary representations of SU(2, 2|4) via the oscillator method

The centrally extended symmetry supergroup of the compactification of type IIB superstring theory over the 5-sphere is SU(2, 2|4), which has the even subgroup SU(2, 2)×SU(4)× U(1) [65]. The generator of the abelian factor U(1) in the even subgroup of SU(2, 2|4) commutes with all the generators and acts like a central charge. Therefore, $\mathfrak{su}(2,2|4)$ is not a simple Lie superalgebra. By factoring out this abelian ideal, one obtains a simple Lie superalgebra, denoted as $\mathfrak{psu}(2,2|4)$, whose even subsuperalgebra is simply $\mathfrak{su}(2,2)\oplus\mathfrak{su}(4)$. Both SU(2, 2|4) and PSU(2, 2|4) have an outer automorphism group $U(1)_Y$ that can be identified with a U(1) subgroup of the $SU(1,1)_{\text{global}} \times U(1)_{\text{local}}$ symmetry of IIB supergravity in ten dimensions [65, 71, 72].

The superalgebra $\mathfrak{su}(2,2|4)$ has a 3-graded decomposition (as in equations (2.1)–(2.3)) with respect to its compact subsuperalgebra $\mathfrak{g}^{(0)} = \mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2) \oplus \mathfrak{u}(1)$, where each $\mathfrak{su}(2|2)$ has an even subalgebra $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ such that one $\mathfrak{su}(2)$ comes from $\mathfrak{su}(2,2)$ and the other from $\mathfrak{su}(4)$.

The Lie superalgebra $\mathfrak{su}(2,2|4)$ can be realized in terms of bilinear combinations of bosonic and fermionic annihilation and creation operators ξ_A, η_M and ξ^A $(= \xi_A{}^\dagger)$, η^M $(= \eta_M{}^\dagger)$, which transform covariantly and contravariantly, respectively, under the two SU$(2|2)$ subsupergroups of SU$(2,2|4)$:

$$\begin{aligned} \xi_A(K) &= \begin{pmatrix} a_i(K) \\ \alpha_\gamma(K) \end{pmatrix}, \qquad & \xi^A(K) &= \begin{pmatrix} a^i(K) \\ \alpha^\gamma(K) \end{pmatrix} \\ \eta_M(K) &= \begin{pmatrix} b_r(K) \\ \beta_\mu(K) \end{pmatrix}, \qquad & \eta^M(K) &= \begin{pmatrix} b^r(K) \\ \beta^\mu(K) \end{pmatrix} \end{aligned} \tag{2.18}$$

with $i = 1,2$; $\gamma = 1,2$; $r = 1,2$; $\mu = 1,2$ and satisfy the super-commutation relations

$$\left[\xi_A(K), \xi^B(L)\right\} = \delta_A^B \delta_{KL}\,, \qquad \left[\eta_M(K), \eta^N(L)\right\} = \delta_M^N \delta_{KL}\,, \tag{2.19}$$

while all the others vanish.

The generators of SU$(2,2|4)$ are given in terms of the above super-oscillators as

$$\begin{aligned} \mathfrak{g}^{(-1)} &= \vec{\xi}_A \cdot \vec{\eta}_M \\ \mathfrak{g}^{(+1)} &= \vec{\eta}^M \cdot \vec{\xi}^A \\ \mathfrak{g}^{(0)} &= \vec{\xi}^A \cdot \vec{\xi}_B \oplus \vec{\eta}_N \cdot \vec{\eta}^M\,. \end{aligned} \tag{2.20}$$

Note that by restricting oneself to bilinears involving only bosonic or fermionic oscillators, one obtains the realizations of SU$(2,2)$ or SU(4) given above, respectively.

Half of the supersymmetry generators belong to the subspace $\mathfrak{g}^{(0)}$ of $\mathfrak{g}$ and the rest belong to the subspace $\mathfrak{g}^{(-1)} \oplus \mathfrak{g}^{(+1)}$. The supersymmetry generators belonging to the $\mathfrak{g}^{(-1)} \oplus \mathfrak{g}^{(+1)}$ subspace obviously close into $\mathfrak{g}^{(0)}$ as follows:

$$\begin{aligned} \left\{Q_{i\mu}, Q^{\nu j}\right\} &= \left\{\vec{a}_i \cdot \vec{\beta}_\mu, \vec{\beta}^\nu \cdot \vec{a}^j\right\} = \delta_\mu^\nu L^j{}_i - \delta_i^j S^\nu{}_\mu + \delta_i^j \delta_\mu^\nu D \\ \left\{Q_{\gamma r}, Q^{s\delta}\right\} &= \left\{\vec{\alpha}_\gamma \cdot \vec{b}_r, \vec{b}^s \cdot \vec{\alpha}^\delta\right\} = \delta_\gamma^\delta R^s{}_r - \delta_r^s M^\delta{}_\gamma + \delta_r^s \delta_\gamma^\delta F\,, \end{aligned} \tag{2.21}$$

where D and F are defined as

$$\begin{aligned} D &= \frac{1}{2}(N_a - N_\beta) + P \\ F &= \frac{1}{2}(N_b - N_\alpha) + P\,. \end{aligned} \tag{2.22}$$

The supersymmetry generators belonging to $\mathfrak{g}^{(0)}$ subspace satisfy the anti-commutation relations:

$$\begin{aligned} \left\{Q^i{}_\gamma, Q^\delta{}_j\right\} &= \left\{\vec{a}^i \cdot \vec{\alpha}_\gamma, \vec{\alpha}^\delta \cdot \vec{a}_j\right\} = \delta^\delta_\gamma L^i{}_j + \delta^i_j M^\delta{}_\gamma + \delta^i_j \delta^\delta_\gamma (E - F) \\ \left\{Q^r{}_\mu, Q^\nu{}_s\right\} &= \left\{\vec{b}^r \cdot \vec{\beta}_\mu, \vec{\beta}^\nu \cdot \vec{b}_s\right\} = \delta^\nu_\mu R^r{}_s + \delta^r_s S^\nu{}_\mu + \delta^r_s \delta^\nu_\mu (E - D)\ . \end{aligned} \tag{2.23}$$

Note that only three of the four U(1) charges E, C, D, F are linearly independent and the linear combinations $(E + C)$, $(E - F)$ and $(E - D)$ do not depend on the number of colors P when expressed in normal ordered form.

Given the above super-oscillator realization, one can easily construct the positive energy UIRs of $SU(2,2|4)$ using the procedure outlined in the beginning of this section — by choosing a set of states $|\Omega\rangle$ in the Fock space that transforms irreducibly under $SU(2|2) \times SU(2|2) \times U(1)$ and is annihilated by $\mathfrak{g}^{(-1)}$, and then by repeatedly acting with the generators of $\mathfrak{g}^{(+1)}$. As mentioned in the beginning, the irreducibility of the resulting positive energy UIRs of $SU(2,2|4)$ follows from the irreducibility of the "lowest representation" $|\Omega\rangle$ under $SU(2|2) \times SU(2|2) \times U(1)$.

By restricting ourselves to the generators involving purely bosonic or purely fermionic oscillators, we recover the subalgebra $\mathfrak{su}(2,2)$ or $\mathfrak{su}(4)$, respectively, and the above construction then yields their UIRs.

Finally, we note that these positive energy UIRs of $SU(2,2|4)$ decompose, in general, into a direct sum of finitely many positive energy UIRs of $SU(2,2)$ transforming in certain representations of the internal symmetry group $SU(4)$.

The spectrum of IIB supergravity over $AdS_5 \times S^5$, as obtained in [65], is given in table 1.

$SU(2)_{j_1} \times SU(2)_{k_1}$ $\times SU(2)_{j_2} \times SU(2)_{k_2}$ Young tableau	$SO(4) =$ $SU(2)_{j_1} \times$ $SU(2)_{j_2}$ labels	AdS Energy E	Field of UIR of $U(2,2\|4)$	$SU(4)$ Dynkin labels	$U(1)_Y$ quantum number
$p \geqslant 1$					
$\|0\rangle$	$(0,0)$	P	$\phi^{(1)}$	$(0,P,0)$	0
$\|\square,1,1,\square\rangle$	$(\frac{1}{2},0)$	$P+\frac{1}{2}$	$\lambda_+^{(1)}$	$(0,P-1,1)$	$\frac{1}{2}$
$\|1,\square,\square,1\rangle$	$(0,\frac{1}{2})$	$P+\frac{1}{2}$	$\lambda_-^{(1)}$	$(1,P-1,0)$	$-\frac{1}{2}$
$\left\|\square\square,1,1,\begin{smallmatrix}\square\\\square\end{smallmatrix}\right\rangle$	$(1,0)$	$P+1$	$A_{\mu\nu}^{(1)}$	$(0,P-1,0)$	1
$\left\|1,\begin{smallmatrix}\square\\\square\end{smallmatrix},\square\square,1\right\rangle$	$(0,1)$	$P+1$	$\tilde{A}_{\mu\nu}^{(1)}$	$(0,P-1,0)$	-1
$p \geqslant 2$					
$\left\|\begin{smallmatrix}\square\\\square\end{smallmatrix},1,1,\square\square\right\rangle$	$(0,0)$	$P+1$	$\phi^{(2)}$	$(0,P-2,2)$	1

Table 1: Continued.

$\left\|1,\yng(2),\yng(1,1),1\right\rangle$	$(0,0)$	$P+1$	$\bar{\phi}^{(2)}$	$(2,P-2,0)$	-1
$\left\|\yng(2,2),1,1,\yng(2,2)\right\rangle$	$(0,0)$	$P+2$	$\phi^{(3)}$	$(0,P-2,0)$	2
$\left\|1,\yng(2,2),\yng(2,2),1\right\rangle$	$(0,0)$	$P+2$	$\bar{\phi}^{(3)}$	$(0,P-2,0)$	-2
$\left\|\yng(2,1),1,1,\yng(2,1)\right\rangle$	$(\frac{1}{2},0)$	$P+\frac{3}{2}$	$\lambda_+^{(2)}$	$(0,P-2,1)$	$\frac{3}{2}$
$\left\|1,\yng(2,1),\yng(2,1),1\right\rangle$	$(0,\frac{1}{2})$	$P+\frac{3}{2}$	$\lambda_-^{(2)}$	$(1,P-2,0)$	$-\frac{3}{2}$
$\left\|\yng(1),\yng(1),\yng(1),\yng(1)\right\rangle$	$(\frac{1}{2},\frac{1}{2})$	$P+1$	$A_\mu^{(1)}$	$(1,P-2,1)$	0
$\left\|\yng(2),\yng(1),\yng(1),\yng(1,1)\right\rangle$	$(1,\frac{1}{2})$	$P+\frac{3}{2}$	$\psi_{+\mu}^{(1)}$	$(1,P-2,0)$	$\frac{1}{2}$
$\left\|\yng(1),\yng(1,1),\yng(2),\yng(1)\right\rangle$	$(\frac{1}{2},1)$	$P+\frac{3}{2}$	$\psi_{-\mu}^{(1)}$	$(0,P-2,1)$	$-\frac{1}{2}$
$\left\|\yng(2),\yng(1,1),\yng(2),\yng(1,1)\right\rangle$	$(1,1)$	$P+2$	$h_{\mu\nu}$	$(0,P-2,0)$	0
$p \geqslant 3$					
$\left\|\yng(1),\yng(2),\yng(1,1),\yng(1)\right\rangle$	$(\frac{1}{2},0)$	$P+\frac{3}{2}$	$\lambda_+^{(3)}$	$(2,P-3,1)$	$-\frac{1}{2}$
$\left\|\yng(1,1),\yng(1),\yng(1),\yng(2)\right\rangle$	$(0,\frac{1}{2})$	$P+\frac{3}{2}$	$\lambda_-^{(3)}$	$(1,P-3,2)$	$\frac{1}{2}$
$\left\|\yng(1),\yng(2,2),\yng(2,2),\yng(1)\right\rangle$	$(\frac{1}{2},0)$	$P+\frac{5}{2}$	$\lambda_+^{(4)}$	$(0,P-3,1)$	$-\frac{3}{2}$
$\left\|\yng(2,2),\yng(1),\yng(1),\yng(2,2)\right\rangle$	$(0,\frac{1}{2})$	$P+\frac{5}{2}$	$\lambda_-^{(4)}$	$(1,P-3,0)$	$\frac{3}{2}$
$\left\|\yng(2,1),\yng(1),\yng(1),\yng(2,1)\right\rangle$	$(\frac{1}{2},\frac{1}{2})$	$P+2$	$A_\mu^{(2)}$	$(1,P-3,1)$	1
$\left\|\yng(1),\yng(2,1),\yng(2,1),\yng(1)\right\rangle$	$(\frac{1}{2},\frac{1}{2})$	$P+2$	$\bar{A}_\mu^{(2)}$	$(1,P-3,1)$	-1
$\left\|\yng(2),\yng(2),\yng(1,1),\yng(1,1)\right\rangle$	$(1,0)$	$P+2$	$A_{\mu\nu}^{(2)}$	$(2,P-3,0)$	0
$\left\|\yng(1,1),\yng(1,1),\yng(2),\yng(2)\right\rangle$	$(0,1)$	$P+2$	$\tilde{A}_{\mu\nu}^{(2)}$	$(0,P-3,2)$	0
$\left\|\yng(2),\yng(2,2),\yng(2,2),\yng(1,1)\right\rangle$	$(1,0)$	$P+3$	$A_{\mu\nu}^{(3)}$	$(0,P-3,0)$	-1
$\left\|\yng(2,2),\yng(1,1),\yng(2),\yng(2,2)\right\rangle$	$(0,1)$	$P+3$	$\tilde{A}_{\mu\nu}^{(3)}$	$(0,P-3,0)$	1
$\left\|\yng(2),\yng(2,1),\yng(2,1),\yng(1,1)\right\rangle$	$(1,\frac{1}{2})$	$P+\frac{5}{2}$	$\psi_{+\mu}^{(2)}$	$(1,P-3,0)$	$-\frac{1}{2}$
$\left\|\yng(2,1),\yng(1,1),\yng(2),\yng(2,1)\right\rangle$	$(\frac{1}{2},1)$	$P+\frac{5}{2}$	$\psi_{-\mu}^{(2)}$	$(0,P-3,1)$	$\frac{1}{2}$
$p \geqslant 4$					
$\left\|\yng(1,1),\yng(2),\yng(1,1),\yng(2)\right\rangle$	$(0,0)$	$P+2$	$\phi^{(4)}$	$(2,P-4,2)$	0
$\left\|\yng(1,1),\yng(2,2),\yng(2,2),\yng(2)\right\rangle$	$(0,0)$	$P+3$	$\phi^{(5)}$	$(0,P-4,2)$	-1
$\left\|\yng(2,2),\yng(2),\yng(1,1),\yng(2,2)\right\rangle$	$(0,0)$	$P+3$	$\bar{\phi}^{(5)}$	$(2,P-4,0)$	1
$\left\|\yng(2,2),\yng(2,2),\yng(2,2),\yng(2,2)\right\rangle$	$(0,0)$	$P+4$	$\phi^{(6)}$	$(0,P-4,0)$	0

Table 1: Continued.

$\left\vert \yng(2,1), \yng(2), \yng(1,1), \yng(2,1) \right\rangle$	$(\frac{1}{2}, 0)$	$P+\frac{5}{2}$	$\lambda_+^{(5)}$	$(2, P-4, 1)$	$\frac{1}{2}$
$\left\vert \yng(1,1), \yng(2,1), \yng(2,1), \yng(2) \right\rangle$	$(0, \frac{1}{2})$	$P+\frac{5}{2}$	$\lambda_-^{(5)}$	$(1, P-4, 2)$	$-\frac{1}{2}$
$\left\vert \yng(2,1), \yng(2,2), \yng(2,2), \yng(2,1) \right\rangle$	$(\frac{1}{2}, 0)$	$P+\frac{7}{2}$	$\lambda_+^{(6)}$	$(0, P-4, 1)$	$-\frac{1}{2}$
$\left\vert \yng(2,2), \yng(2,1), \yng(2,1), \yng(2,2) \right\rangle$	$(0, \frac{1}{2})$	$P+\frac{7}{2}$	$\lambda_-^{(6)}$	$(1, P-4, 0)$	$\frac{1}{2}$
$\left\vert \yng(2,1), \yng(2,1), \yng(2,1), \yng(2,1) \right\rangle$	$(\frac{1}{2}, \frac{1}{2})$	$P+3$	$A_\mu^{(3)}$	$(1, P-4, 1)$	0

Table 1: The spectrum of the IIB supergravity compactified on S^5. The states listed above for a given P, together with their AdS excitations form a UIR of $SU(2,2|4)$. E is the AdS energy and $SU(2)_{j_1} \times SU(2)_{j_2} = SU(2)_L \times SU(2)_R$. The lowest weight vector for each supermultiplet is the Fock vacuum. The first column gives the $SU(2)^{\otimes 4}$ transformation properties of the states obtained by the action of supersymmetry generators in $\mathfrak{g}^{(+1)}$ space on the vacuum. $P = 1$ supermultiplet is the Yang-Mills doubleton supermultiplet and decouples from the spectrum.

2.2 Oscillator construction of the positive energy representations of $OSp(8^*|4)$

The symmetry supergroup of the 11-dimensional supergravity compactified to AdS space on S^4 is $OSp(8^*|4)$, whose even subgroup is $SO^*(8) \times USp(4)$. A detailed construction of UIRs of this supergroup using the oscillator method is given in [69, 74, 75]. Here, we give only a brief summary of the results and avoid repeating the general details that have already been mentioned in the above section on $SU(2,2|4)$.

2.2.1 $SO^*(8) \approx SO(6,2)$ representations via the oscillator method

The non-compact group $SO^*(8)$, which is isomorphic to $SO(6,2)$ is the conformal group in $d = 6$ as well as the anti-de Sitter group in $d = 7$.

The 3-grading (as in equations (2.1)–(2.3)) of the Lie algebra of $\mathfrak{so}^*(8) \approx \mathfrak{so}(6,2)$ is defined with respect to its maximal compact subalgebra $\mathfrak{su}(4) \oplus \mathfrak{u}(1)$.

As was done in the previous case, to construct the positive-energy UIRs of $SO^*(8)$, one introduces an arbitrary number P pairs of bosonic annihilation and creation operators $a_i(K)$, $b_i(K)$ and $a^i(K) = a_i(K)^\dagger$, $b^i(K) = b_i(K)^\dagger$ ($i = 1, \ldots, 4$; $K = 1, \ldots, P$), which transform as $\overline{\mathbf{4}}$ and $\mathbf{4}$ representations of the maximal compact subgroup $U(4) = SU(4) \times U(1)$ and satisfy the usual canonical commutation relations:

$$\left[a_i(K), a^j(L)\right] = \delta_i^j \delta_{KL}\,, \qquad \left[b_i(K), b^j(L)\right] = \delta_i^j \delta_{KL}\,, \tag{2.24}$$

while all the other commutators vanish. The vacuum state $|0\rangle$ is defined by:

$$a_i(K)\,|0\rangle = 0 = b_i(K)\,|0\rangle \tag{2.25}$$

for all $i = 1, \ldots, 4$; $K = 1, \ldots, P$.

The Lie algebra of $\mathfrak{so}^*(8)$ is now realized as bilinears of these bosonic oscillators in the following manner:

$$\begin{aligned} M^i{}_j &= \vec{a}^i \cdot \vec{a}_j + \vec{b}_j \cdot \vec{b}^i \\ A_{ij} &= \vec{a}_i \cdot \vec{b}_j - \vec{a}_j \cdot \vec{b}_i \\ A^{ij} &= \vec{a}^i \cdot \vec{b}^j - \vec{a}^j \cdot \vec{b}^i\,. \end{aligned} \tag{2.26}$$

The generators of $\mathfrak{g}^{(-1)}$ and $\mathfrak{g}^{(+1)}$ subspaces commute to give

$$\left[A_{ij}, A^{kl}\right] = \delta_i^k M^l{}_j - \delta_i^l M^k{}_j - \delta_j^k M^l{}_i + \delta_j^l M^k{}_i \,. \tag{2.27}$$

$M^i{}_j$ form the Lie algebra of $\mathfrak{u}(4)$, while A_{ij} and A^{ij}, both transforming as **6** of SU(4) with opposite charges under the $\mathfrak{u}(1)$ generator $M^i{}_i$, extend this $\mathfrak{u}(4)$ to the Lie algebra of $\mathfrak{so}^*(8)$. The $\mathfrak{u}(1)$ charge $M^i{}_i$ gives the AdS energy

$$E = \frac{1}{2} M^i{}_i = \frac{1}{2} N_B + 2P \,, \tag{2.28}$$

where $N_B = \vec{a}^i \cdot \vec{a}_i + \vec{b}^i \cdot \vec{b}_i$ is the bosonic number operator.

Now the lowest weight UIRs of $\mathrm{SO}^*(8) \approx \mathrm{SO}(6,2)$ can be constructed, as outlined in the general discussion in the beginning of section 2, by choosing a set of states $|\Omega\rangle$ that transforms irreducibly under the maximal compact subgroup $\mathrm{SU}(4) \times \mathrm{U}(1)$ and is annihilated by all the generators in $\mathfrak{g}^{(-1)}$ subspace. These UIRs, constructed by acting on $|\Omega\rangle$ repeatedly with the elements of $\mathfrak{g}^{(+1)}$ (as in equation (2.4)), are uniquely determined by this lowest weight vector $|\Omega\rangle$ and can be identified with fields in AdS_7 or conformal fields in $d = 6$ [69, 74, 75].

Finally, we should mention that the doubleton representations of $\mathrm{SO}^*(8)$ constructed this way (by taking only one pair of oscillators, $P = 1$) do not have a Poincaré limit in $d = 7$. The Poincaré mass operator in $d = 6$ vanishes identically for these representations and therefore they correspond to massless conformal fields in $d = 6$. The tensoring of two copies of these doubletons (taking $P = 2$) produces massless representations of AdS_7, but in the CFT_6 sense they correspond to massive conformal fields. Tensoring more than two copies ($P > 2$) leads to representations that are massive both in the AdS_7 and CFT_6 sense [69, 74, 75].

2.2.2 USp(4) $\approx$ SO(5) representations via the oscillator method

To construct all the UIRs of $\mathrm{USp}(4) \approx \mathrm{SO}(5)$, we introduce P pairs of fermionic annihilation and creation operators $\alpha_\kappa(K)$, $\beta_\kappa(K)$ and $\alpha^\kappa(K) = \alpha_\kappa(K)^\dagger$, $\beta^\kappa(K) = \beta_\kappa(K)^\dagger$ ($\kappa = 1, 2$; $K = 1, \dots, P$), which transform in the doublet of SU(2) and satisfy the usual anti-commutation relations:

$$\{\alpha_\kappa(K), \alpha^\rho(L)\} = \delta_\kappa^\rho \delta_{KL} \,, \qquad \{\beta_\kappa(K), \beta^\rho(L)\} = \delta_\kappa^\rho \delta_{KL} \,, \tag{2.29}$$

while all the other anti-commutators vanish. Once again, the vacuum state is annihilated by all the annihilation operators:

$$\alpha_\kappa(K)\,|0\rangle = 0 = \beta_\kappa(K)\,|0\rangle \tag{2.30}$$

for all $\kappa = 1, 2$; $K = 1, \dots, P$.

The Lie algebra of $\mathfrak{so}(5)$ is now realized as bilinears of these fermionic oscillators:

$$\begin{aligned} M^\kappa{}_\rho &= \vec{\alpha}^\kappa \cdot \vec{\alpha}_\rho - \vec{\beta}_\rho \cdot \vec{\beta}^\kappa \\ A_{\kappa\rho} &= \vec{\alpha}_\kappa \cdot \vec{\beta}_\rho + \vec{\alpha}_\rho \cdot \vec{\beta}_\kappa \\ A^{\kappa\rho} &= \vec{\alpha}^\kappa \cdot \vec{\beta}^\rho + \vec{\alpha}^\rho \cdot \vec{\beta}^\kappa \,. \end{aligned} \tag{2.31}$$

Thus, the generators of $\mathfrak{g}^{(-1)}$ and $\mathfrak{g}^{(+1)}$ subspaces satisfy

$$\left[A_{\kappa\rho}, A^{\lambda\sigma}\right] = \delta^{\lambda}_{\kappa} M^{\sigma}{}_{\rho} + \delta^{\sigma}_{\kappa} M^{\lambda}{}_{\rho} + \delta^{\lambda}_{\rho} M^{\sigma}{}_{\kappa} + \delta^{\sigma}_{\rho} M^{\lambda}{}_{\kappa} \,. \tag{2.32}$$

$M^{\kappa}{}_{\rho}$ generate the Lie algebra of $\mathfrak{u}(2)$ and $A_{\kappa\rho}$ and $A^{\kappa\rho}$ extend it to that of $\mathfrak{so}(5)$. The $\mathfrak{u}(1)$ charge with respect to which this 3-grading is defined is

$$C = \frac{1}{2} M^{\kappa}{}_{\kappa} = \frac{1}{2} N_F - P \,, \tag{2.33}$$

where $N_F = \vec{\alpha}^{\kappa} \cdot \vec{\alpha}_{\kappa} + \vec{\beta}^{\kappa} \cdot \vec{\beta}_{\kappa}$ is the fermionic number operator.

The choice of the lowest representations $|\Omega\rangle$ (that transform irreducibly under U(2) and are annihilated by the generators of $\mathfrak{g}^{(-1)}$ subspace) and the construction of the representations of USp(4) $\approx$ SO(5) can now be done analogous to the previous section. Just as in the case of SU(4) above (section 2.1.2), because of the fermionic nature of the oscillators, equation (2.4) produces only finite-dimensional representations.

2.2.3 Unitary representations of OSp($8^*|4$) via the oscillator method

The superalgebra $\mathfrak{osp}(8^*|4)$ has a 3-grading with respect to its compact subsuperalgebra $\mathfrak{u}(4|2)$, which has an even part $\mathfrak{u}(4) \oplus \mathfrak{u}(2)$.

Thus, to construct the UIRs of OSp($8^*|4$), one defines the U($4|2$) covariant super-oscillators as follows:

$$\begin{aligned}
\xi_A(K) &= \begin{pmatrix} a_i(K) \\ \alpha_\kappa(K) \end{pmatrix}, \qquad & \xi^A(K) = \xi_A(K)^\dagger &= \begin{pmatrix} a^i(K) \\ \alpha^\kappa(K) \end{pmatrix} \\
\eta_A(K) &= \begin{pmatrix} b_i(K) \\ \beta_\kappa(K) \end{pmatrix}, \qquad & \eta^A(K) = \eta_A(K)^\dagger &= \begin{pmatrix} b^i(K) \\ \beta^\kappa(K) \end{pmatrix}
\end{aligned} \tag{2.34}$$

where $i = 1, \ldots, 4$; $\kappa = 1, 2$; $K = 1, \ldots, P$. They satisfy the super-commutation relations:

$$\left[\xi_A(K), \xi^B(L)\right\} = \delta^B_A \delta_{KL} \,, \qquad \left[\eta_A(K), \eta^B(L)\right\} = \delta^B_A \delta_{KL} \,. \tag{2.35}$$

Now, the Lie superalgebra $\mathfrak{osp}(8^*|4)$ can be realized in terms of the following bilinears:

$$\begin{aligned}
M^A{}_B &= \vec{\xi}^A \cdot \vec{\xi}_B + (-1)^{(\deg A)(\deg B)} \vec{\eta}_B \cdot \vec{\eta}^A \\
A_{AB} &= \vec{\xi}_A \cdot \vec{\eta}_B - \vec{\eta}_A \cdot \vec{\xi}_B \\
A^{AB} &= \vec{\eta}^B \cdot \vec{\xi}^A - \vec{\xi}^B \cdot \vec{\eta}^A \,,
\end{aligned} \tag{2.36}$$

where $\deg A = 0$ ($\deg A = 1$) if A is a bosonic (fermionic) index. Clearly, $M^A{}_B$ generate the $\mathfrak{g}^{(0)}$ subspace $\mathfrak{u}(4|2)$, while A_{AB} and A^{AB}, which correspond to $\mathfrak{g}^{(-1)}$ and $\mathfrak{g}^{(+1)}$ subspaces respectively, extend this to the full $\mathfrak{osp}(8^*|4)$ superalgebra. It is worth noting that the above 3-grading is defined with respect to the abelian $\mathfrak{u}(1)$ charge of $\mathfrak{u}(4|2)$:

$$E + C = \frac{1}{2} M^A{}_A = \frac{1}{2}\left(N_B + N_F\right) + P \,. \tag{2.37}$$

Once again, half of the supersymmetry generators belong to the $\mathfrak{g}^{(0)}$ subspace ($Q^{i}{}_{\kappa} \oplus Q^{\kappa}{}_{i}$) and the rest belong to the $\mathfrak{g}^{(-1)} \oplus \mathfrak{g}^{(+1)}$ subspace ($Q_{i\kappa} \oplus Q^{i\kappa}$):

$$\begin{aligned} \left\{ Q_{i\kappa}, Q^{j\rho} \right\} &= \delta^{\rho}_{\kappa} M^{j}{}_{i} - \delta^{j}_{i} M^{\rho}{}_{\kappa} \\ \left\{ Q^{i}{}_{\kappa}, Q^{\rho}{}_{j} \right\} &= \delta^{\rho}_{\kappa} M^{i}{}_{j} + \delta^{i}_{j} M^{\rho}{}_{\kappa} \,. \end{aligned} \tag{2.38}$$

Given this super-oscillator realization, once again, one can easily construct the positive energy UIRs of OSp$(8^*|4)$ by first choosing a set of states $|\Omega\rangle$ in the Fock space that transforms irreducibly under U$(4|2)$ and is annihilated by all the generators of $\mathfrak{g}^{(-1)}$ subspace, and then repeatedly acting with the generators of $\mathfrak{g}^{(+1)}$.

The spectrum of 11-dimensional supergravity over AdS$_7 \times$ S^4, as given in [69], is reproduced in table 2.

SU(4) × SU(2) Young tableau	SU(4) Dynkin labels	USp(4) Dynkin labels	AdS energy E	Field in $d = 7$
$P \geqslant 1$				
$\vert 0,0\rangle$	$(0,0,0)$	$(0,P)$	$2P$	scalar
$\vert \square,\square\rangle$	$(1,0,0)$	$(1,P-1)$	$2P+\frac{1}{2}$	spinor
$\left\vert \yng(2),\yng(1,1)\right\rangle$	$(2,0,0)$	$(0,P-1)$	$2P+1$	$\sqrt{a_{\alpha\beta\gamma}}$
$P \geqslant 2$				
$\left\vert \yng(1,1),\yng(2)\right\rangle$	$(0,1,0)$	$(2,P-2)$	$2P+1$	vector
$\left\vert \yng(2,1),\yng(2,1)\right\rangle$	$(1,1,0)$	$(1,P-2)$	$2P+\frac{3}{2}$	gravitino
$\left\vert \yng(2,2),\yng(2,2)\right\rangle$	$(0,2,0)$	$(0,P-2)$	$2P+2$	graviton
$P \geqslant 3$				
$\left\vert \yng(1,1,1),\yng(3)\right\rangle$	$(0,0,1)$	$(3,P-3)$	$2P+\frac{3}{2}$	spinor
$\left\vert \yng(2,1,1),\yng(3,1)\right\rangle$	$(1,0,1)$	$(2,P-3)$	$2P+2$	$a_{\alpha\beta}$
$\left\vert \yng(2,2,1),\yng(3,2)\right\rangle$	$(0,1,1)$	$(1,P-3)$	$2P+\frac{5}{2}$	gravitino
$\left\vert \yng(2,2,2),\yng(3,3)\right\rangle$	$(0,0,2)$	$(0,P-3)$	$2P+3$	$\sqrt{a_{\alpha\beta\gamma}}$
$P \geqslant 4$				
$\left\vert \yng(1,1,1,1),\yng(4)\right\rangle$	$(0,0,0)$	$(4,P-4)$	$2P+2$	scalar

Table 2: Continued.

JHEP10(2002)007

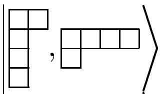
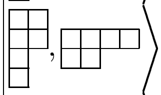
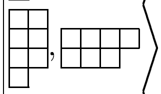
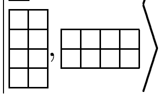
	$(1,0,0)$	$(3,P-4)$	$2P+\frac{5}{2}$	spinor
	$(0,1,0)$	$(2,P-4)$	$2P+3$	vector
	$(0,0,1)$	$(1,P-4)$	$2P+\frac{7}{2}$	spinor
	$(0,0,0)$	$(0,P-4)$	$2P+4$	scalar

Table 2: The spectrum of 11-dimensional supergravity compactified on S^4. The states listed above for a given P, together with their AdS excitations form a UIR of $OSp(8^*|4)$. The lowest weight vector for each UIR is the Fock vacuum. The first column gives the $SU(4) \times SU(2)$ transformation properties of the states obtained by the action of supersymmetry generators in $\mathfrak{g}^{(+1)}$ space on the vacuum. Once again, $P = 1$ doubleton supermultiplet decouples from the spectrum.

2.3 Oscillator construction of the positive energy representations of $OSp(8|4,\mathbb{R})$

The compactification of 11-dimensional supergravity to AdS_4 space on S^7 has an $OSp(8|4,\mathbb{R})$ symmetry. This supergroup $OSp(8|4,\mathbb{R})$ has an even subgroup $SO(8) \times Sp(4,\mathbb{R})$ and in this section, we give a brief outline of the construction of its positive energy UIRs following [70].

2.3.1 $SO(3,2) \approx Sp(4,\mathbb{R})$ representations via the oscillator method

The non-compact group $SO(3,2) \approx Sp(4,\mathbb{R})$ is the AdS group in four dimensions.

The 3-grading (equations (2.1)-(2.3)) of the Lie algebra of $\mathfrak{sp}(4,\mathbb{R}) \approx \mathfrak{so}(3,2)$ is defined with respect to its maximal compact subalgebra $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$.

Therefore, to construct the positive-energy UIRs of $Sp(4,\mathbb{R})$, one introduces an arbitrary number of bosonic annihilation and creation operators. However, unlike in the previous case of $SO^*(8)$ (section 2.2.1), where one has to choose an even number of oscillators $a_i(1), \ldots, a_i(P)$; $b_i(1), \ldots, b_i(P)$, here one also has the freedom of choosing an odd number.

Choosing an even number of oscillators constitutes taking $n = 2P$ annihilation operators $a_i(K)$, $b_i(K)$ $(K = 1, \ldots, P)$ and their hermitian conjugate creation operators $a^i(K)$, $b^i(K)$, which transform covariantly and contravariantly, respectively, with respect to $SU(2)$ $(i = 1, 2)$. On the other hand, an odd number $n = 2P + 1$ of oscillators can be chosen by taking an extra oscillator c_i and its conjugate c^i in addition to the above $2P$ oscillators. They satisfy the commutation relations:

$$\left[a_i(K), a^j(L)\right] = \delta_i^j \delta_{KL}\,, \qquad \left[b_i(K), b^j(L)\right] = \delta_i^j \delta_{KL}\,, \qquad \left[c_i, c^j\right] = \delta_i^j \quad \text{(if present)}\,, \tag{2.39}$$

while all the other commutators vanish. The vacuum state $|0\rangle$ is annihilated by all $a_i(K)$ and $b_i(K)$ $(i = 1, 2;\ K = 1, \ldots, P)$ as well as, if present, by all c_i $(i = 1, 2)$.

Thus, the Lie algebra $\mathfrak{sp}(4,\mathbb{R})$ is realized as bilinears of these bosonic oscillators in the following manner:

$$\begin{aligned} M^i{}_j &= \vec{a}^i \cdot \vec{a}_j + \vec{b}_j \cdot \vec{b}^i + \epsilon \frac{1}{2}\left(c^i c_j + c_j c^i\right) \\ A_{ij} &= \vec{a}_i \cdot \vec{b}_j + \vec{a}_j \cdot \vec{b}_i + \epsilon c_i c_j \\ A^{ij} &= \vec{a}^i \cdot \vec{b}^j + \vec{a}^j \cdot \vec{b}^i + \epsilon c^i c^j \,, \end{aligned} \tag{2.40}$$

where $\epsilon = 0$ ($\epsilon = 1$) if the number of oscillators n is even (odd). The generators of $\mathfrak{g}^{(-1)}$ and $\mathfrak{g}^{(+1)}$ subspaces satisfy

$$\left[A_{ij}, A^{kl}\right] = \delta_i^k M^l{}_j + \delta_i^l M^k{}_j + \delta_j^k M^l{}_i + \delta_j^l M^k{}_i \,. \tag{2.41}$$

$M^i{}_j$ form the maximal compact subalgebra $\mathfrak{u}(2)$ of $\mathfrak{sp}(4,\mathbb{R})$, while A_{ij} and A^{ij} extend it to the Lie algebra $\mathfrak{sp}(4,\mathbb{R})$. The $\mathfrak{u}(1)$ charge which defines the 3-grading (equation (2.3)) is $M^i{}_i$ and is given by

$$E = \frac{1}{2} M^i{}_i = \frac{1}{2} N_B + P + \epsilon \frac{1}{2} \,. \tag{2.42}$$

As explained in the previous sections, now the lowest weight UIRs of $\mathrm{Sp}(4,\mathbb{R}) \approx \mathrm{SO}(3,2)$ can be constructed by choosing a set of states $|\Omega\rangle$ that transforms irreducibly under U(2) and is annihilated by all the generators in $\mathfrak{g}^{(-1)}$ subspace. Repeated action on these $|\Omega\rangle$ by the elements of $\mathfrak{g}^{(+1)}$ (equation (2.4)) generates the UIRs.

2.3.2 SO(8) representations via the oscillator method

SO(8) has a 3-grading structure with respect to its subgroup U(4). Therefore, we introduce fermionic annihilation and creation operators that transform as $\bar{\mathbf{4}}$ and $\mathbf{4}$ representations of U(4).

If we choose to have an even number of fermionic oscillators ($n = 2P$), we take $\alpha_\kappa(K)$, $\beta_\kappa(K)$ ($\kappa = 1, \ldots, 4$; $K = 1, \ldots, P$) and their hermitian conjugates $\alpha^\kappa(K)$, $\beta^\kappa(K)$, but if we choose an odd number of oscillators ($n = 2P+1$), we take in addition to the above, another oscillator γ_κ and its conjugate γ^κ. They satisfy the anti-commutation relations:

$$\{\alpha_\kappa(K), \alpha^\rho(L)\} = \delta_\kappa^\rho \delta_{KL} \,, \qquad \{\beta_\kappa(K), \beta^\rho(L)\} = \delta_\kappa^\rho \delta_{KL} \,, \qquad \{\gamma_\kappa, \gamma^\rho\} = \delta_\kappa^\rho \quad \text{(if present)} \,, \tag{2.43}$$

while all the other anti-commutators vanish. Once again, the vacuum state $|0\rangle$ is annihilated by all the annihilation operators $\alpha_\kappa(K)$, $\beta_\kappa(K)$ and γ_κ (if present) for all values of κ and K.

The Lie algebra of $\mathfrak{so}(8)$ is realized as bilinears of these fermionic oscillators as follows:

$$\begin{aligned} M^\kappa{}_\rho &= \vec{\alpha}^\kappa \cdot \vec{\alpha}_\rho - \vec{\beta}_\rho \cdot \vec{\beta}^\kappa + \epsilon \frac{1}{2}\left(\gamma^\kappa \gamma_\rho - \gamma_\rho \gamma^\kappa\right) \\ A_{\kappa\rho} &= \vec{\alpha}_\kappa \cdot \vec{\beta}_\rho - \vec{\alpha}_\rho \cdot \vec{\beta}_\kappa + \epsilon \gamma_\kappa \gamma_\rho \\ A^{\kappa\rho} &= \vec{\alpha}^\kappa \cdot \vec{\beta}^\rho - \vec{\alpha}^\rho \cdot \vec{\beta}^\kappa + \epsilon \gamma^\kappa \gamma^\rho \,. \end{aligned} \tag{2.44}$$

Thus, the generators of $\mathfrak{g}^{(\pm 1)}$ subspaces satisfy

$$\left[A_{\kappa\rho}, A^{\lambda\sigma}\right] = \delta_\kappa^\lambda M^\sigma{}_\rho - \delta_\kappa^\sigma M^\lambda{}_\rho - \delta_\rho^\lambda M^\sigma{}_\kappa + \delta_\rho^\sigma M^\lambda{}_\kappa \,. \tag{2.45}$$

$M^\kappa{}_\rho$ form the maximal compact subalgebra $\mathfrak{u}(4)$ of $\mathfrak{so}(8)$ and $A_{\kappa\rho}$ and $A^{\kappa\rho}$ extend it to the

Lie algebra $\mathfrak{so}(8)$. The $\mathfrak{u}(1)$ charge with respect to which the 3-grading (equation (2.3)) is defined is

$$C = \frac{1}{2}M^{\kappa}{}_{\kappa} = \frac{1}{2}N_F - 2P - \epsilon \,. \tag{2.46}$$

The choice of the lowest weight vectors $|\Omega\rangle$ (that transform irreducibly under U(4) and are annihilated by $\mathfrak{g}^{(-1)}$ space) and the construction of the representations of SO(8) now proceeds analogous to the previous sections. Once again, because of the fermionic nature of the oscillators, equation (2.4) produces only finite-dimensional representations.

2.3.3 Unitary representations of OSp$(8|4,\mathbb{R})$ via the oscillator method

The superalgebra $\mathfrak{osp}(8|4,\mathbb{R})$ has a 3-grading with respect to its compact subsuperalgebra $\mathfrak{u}(2|4)$, which has an even part $\mathfrak{u}(2)\oplus\mathfrak{u}(4)$.

Thus, to construct the UIRs of OSp$(8|4,\mathbb{R})$, one defines the U$(2|4)$ covariant super-oscillators as follows:

$$\begin{aligned}
\xi_A(K) &= \begin{pmatrix} a_i(K) \\ \alpha_\kappa(K) \end{pmatrix}, & \xi^A(K) = \xi_A(K)^\dagger &= \begin{pmatrix} a^i(K) \\ \alpha^\kappa(K) \end{pmatrix} \\
\eta_A(K) &= \begin{pmatrix} b_i(K) \\ \beta_\kappa(K) \end{pmatrix}, & \eta^A(K) = \eta_A(K)^\dagger &= \begin{pmatrix} b^i(K) \\ \beta^\kappa(K) \end{pmatrix} \\
\zeta_A &= \begin{pmatrix} c_i \\ \gamma_\kappa \end{pmatrix}, & \zeta^A = \zeta_A{}^\dagger &= \begin{pmatrix} c^i \\ \gamma^\kappa \end{pmatrix}
\end{aligned} \tag{2.47}$$

where $i=1,2$; $\kappa=1,\dots,4$; $K=1,\dots,P$. The oscillators a, b, α and β (and c, γ, if present) satisfy the usual (anti)commutation relations and therefore the above super-oscillators satisfy the super-commutation relations:

$$\left[\xi_A(K),\xi^B(L)\right\} = \delta_A^B\delta_{KL}\,, \qquad \left[\eta_A(K),\eta^B(L)\right\} = \delta_A^B\delta_{KL}\,, \qquad \left[\zeta_A,\zeta^B\right\} = \delta_A^B\,. \tag{2.48}$$

Now, in terms of these super-oscillators, the Lie superalgebra $\mathfrak{osp}(8|4,\mathbb{R})$ has the following realization:

$$\begin{aligned}
M^A{}_B &= \vec{\xi}^A\cdot\vec{\xi}_B + (-1)^{(\deg A)(\deg B)}\vec{\eta}_B\cdot\vec{\eta}^A + \epsilon\frac{1}{2}\left(\zeta^A\zeta_B + (-1)^{(\deg A)(\deg B)}\zeta_B\zeta^A\right) \\
A_{AB} &= \vec{\xi}_A\cdot\vec{\eta}_B + \vec{\eta}_A\cdot\vec{\xi}_B + \epsilon\zeta_A\zeta_B \\
A^{AB} &= \vec{\eta}^B\cdot\vec{\xi}^A + \vec{\xi}^B\cdot\vec{\eta}^A + \epsilon\zeta^B\zeta^A\,.
\end{aligned} \tag{2.49}$$

It is easy to see that, $M^A{}_B$ generate the $\mathfrak{g}^{(0)}$ subspace $\mathfrak{u}(2|4)$ and A_{AB} and A^{AB} extend it to the full $\mathfrak{osp}(8|4,\mathbb{R})$ superalgebra. The abelian $\mathfrak{u}(1)$ charge which defines the above 3-grading is

$$E + C = \frac{1}{2}M^A{}_A = \frac{1}{2}\left(N_B + N_F\right) - P - \epsilon\frac{1}{2}. \tag{2.50}$$

Given this super-oscillator realization, once again, one can construct the positive energy UIRs of OSp$(8|4,\mathbb{R})$ by first choosing a set of states $|\Omega\rangle$ in the Fock space that transforms irreducibly under U$(2|4)$ and is annihilated by $\mathfrak{g}^{(-1)}$ and then by repeatedly acting with the generators of $\mathfrak{g}^{(+1)}$.

The spectrum of eleven dimensional supergravity over AdS$_4\times$ S^7, as obtained in [70], is given in table 3.

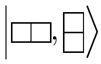
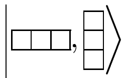
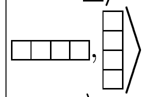
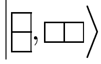
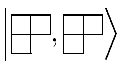
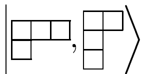
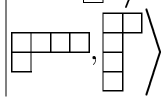
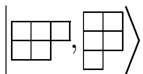
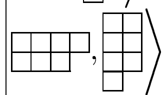
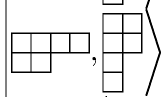
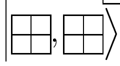
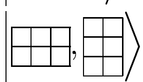
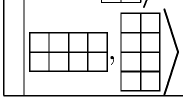
$\mathrm{SU}(2) \times \mathrm{SU}(4)$ Young tableau	Spin and parity	AdS Energy E	$\mathrm{SO}(8)_{G-Z}$ labels	$\mathrm{SO}(8)_D$ labels
$n \geqslant 1$				
$\lvert 0,0\rangle$	0^+	$\frac{n}{2}$	$(n,0,0,0)$	$(n,0,0,0)$
$\lvert\square,\square\rangle$	$\frac{1}{2}$	$\frac{n}{2}+\frac{1}{2}$	$(n-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2})$	$(n-1,0,1,0)$
$n \geqslant 2$				
	1^-	$\frac{n}{2}+1$	$(n-1,1,0,0)$	$(n-2,1,0,0)$
	$\frac{3}{2}$	$\frac{n}{2}+\frac{3}{2}$	$(n-\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$(n-2,0,0,1)$
	2	$\frac{n}{2}+2$	$(n-2,0,0,0)$	$(n-2,0,0,0)$
	0^-	$\frac{n}{2}+1$	$(n-1,1,1,-1)$	$(n-2,0,2,0)$
$n \geqslant 3$				
	$\frac{1}{2}$	$\frac{n}{2}+\frac{3}{2}$	$(n-\frac{3}{2},\frac{3}{2},\frac{1}{2},-\frac{1}{2})$	$(n-3,1,1,0)$
	1^+	$\frac{n}{2}+2$	$(n-2,1,1,0)$	$(n-3,0,1,1)$
	$\frac{3}{2}$	$\frac{n}{2}+\frac{5}{2}$	$(n-\frac{5}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2})$	$(n-3,0,1,0)$
$n \geqslant 4$				
	$\frac{1}{2}$	$\frac{n}{2}+\frac{5}{2}$	$(n-\frac{5}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2})$	$(n-4,1,0,1)$
	$\frac{1}{2}$	$\frac{n}{2}+\frac{7}{2}$	$(n-\frac{7}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$(n-4,0,0,1)$
	1^-	$\frac{n}{2}+3$	$(n-3,1,0,0)$	$(n-4,1,0,0)$
	0^+	$\frac{n}{2}+2$	$(n-2,2,0,0)$	$(n-4,2,0,0)$
	0^-	$\frac{n}{2}+3$	$(n-3,1,1,1)$	$(n-4,0,0,2)$
	0^+	$\frac{n}{2}+4$	$(n-4,0,0,0)$	$(n-4,0,0,0)$

Table 3: The spectrum of the 11-dimensional supergravity compactified on S^7. The states listed above for a given n, together with their AdS excitations form a UIR of $\mathrm{OSp}(8|4,\mathbb{R})$. Note that $n = 2P + \epsilon$. The lowest representation for each supermultiplet is the Fock vacuum. The first column gives the $\mathrm{SU}(2) \times \mathrm{SU}(4)$ transformation properties of the states obtained by the action of supersymmetry generators in $\mathfrak{g}^{(+1)}$ space on the vacuum. $n = 1$ singleton supermultiplet decouples from the spectrum as gauge modes.

3. Contraction of the superalgebras over $\mathrm{AdS}_p \times \mathrm{S}^q$ spaces to maximally supersymmetric PP-wave algebras in the oscillator formalism

Following BFHP [7], we consider the following metric on AdS_p:

$$g_{\mathrm{AdS}_p} = R_{\mathrm{AdS}}{}^2 \left[-d\tau^2 + (\sin\tau)^2 \left(\frac{dr^2}{1+r^2} + r^2 d\Omega_{p-2}{}^2 \right) \right] \tag{3.1}$$

where R_{AdS} is the radius of curvature and $d\Omega_{p-2}{}^2$ is the $(p-2)$-sphere metric. Similarly, we choose the following metric on $\mathrm{S}^{D-p} = \mathrm{S}^q$:

$$g_{\mathrm{S}^q} = R_{\mathrm{S}}{}^2 \left[d\psi^2 + (\sin\psi)^2 d\Omega_{q-1}{}^2 \right] \tag{3.2}$$

where R_{S} is the radius of curvature and $d\Omega_{q-1}{}^2$ is the metric on the equatorial $(q-1)$-sphere. Then the metric on the product space $\mathrm{AdS}_p \times \mathrm{S}^q$ is simply

$$g = g_{\mathrm{AdS}_p} + g_{\mathrm{S}^q} \, . \tag{3.3}$$

As shown by BFHP [7], by defining the coordinates

$$u = \psi + \rho\tau \, , \qquad v = \psi - \rho\tau \qquad \text{where} \quad \rho = \frac{R_{\mathrm{AdS}}}{R_{\mathrm{S}}} \, , \qquad R = R_{\mathrm{S}} \tag{3.4}$$

the metric g on the product space can be written in the form

$$R^{-2} g = du dv + \rho^2 \sin\left(\frac{u-v}{2\rho}\right)^2 \left(\frac{dr^2}{1+r^2} + r^2 d\Omega_{p-2}{}^2 \right) + \sin\left(\frac{u+v}{2}\right)^2 d\Omega_{q-1}{}^2 \, . \tag{3.5}$$

By taking the Penrose limit along the null geodesic parametrized by u, one can then obtain the PP-wave metric on $\mathrm{AdS}_p \times \mathrm{S}^q$ space.

BFHP showed that for $D = 11$, one obtains the metric of a maximally supersymmetric PP-wave if the parameter $\rho = 1/2$ or 2, confirming the earlier results [80, 81]. $\rho = 1/2$ solution corresponds to the PP-wave contraction of the $\mathrm{AdS}_4 \times \mathrm{S}^7$ superalgebra $\mathfrak{osp}(8|4,\mathbb{R})$ with the even subalgebra $\mathfrak{so}(8) \oplus \mathfrak{sp}(4,\mathbb{R})$ and $\rho = 2$ solution represents the PP-wave contraction of the $\mathrm{AdS}_7 \times \mathrm{S}^4$ superalgebra $\mathfrak{osp}(8^*|4)$ with the even subalgebra $\mathfrak{so}^*(8) \oplus \mathfrak{usp}(4)$.

For $D = 10$, there is only one PP-wave solution with maximal supersymmetry, given by $\rho = 1$. The resulting PP-wave algebra is simply the contraction of $\mathfrak{su}(2,2|4)$.

The general oscillator realization of superalgebras corresponds to taking the direct sum of an arbitrary number P of copies (colors) of the singleton or doubleton realizations. Hence the only free parameter available for contraction of the oscillator realization of AdS/Conformal superalgebras to the PP-wave algebras is the number of colors P. When one normal orders all the generators, this parameter P appears explicitly on the right hand side of the commutators of generators in $\mathfrak{g}^{(-1)}$ with those in $\mathfrak{g}^{(+1)}$. More specifically, for the AdS/Conformal superalgebras, the generators that depend explicitly on P after normal ordering are precisely the U(1) generators that determine the 3-grading of the AdS and internal (R-symmetry) subalgebras. For AdS subalgebras, the generator of this U(1) subgroup is the energy operator E. We shall denote the generator that gives the 3-grading in the internal (R-symmetry) subalgebra as $J = -C$, in order to agree with the notation used in BMN [8].

Specifically, for $PSU(2,2|4) \supset SU(2,2) \times SU(4)$, the generators E and J are given by (see equations (2.11), (2.16)):

$$E = \frac{1}{2}N_B + P\,, \qquad J = -\frac{1}{2}N_F + P \tag{3.6}$$

where $N_B = N_a + N_b$ and $N_F = N_\alpha + N_\beta$ are bosonic and fermionic number operators, respectively.

In the geometric realization of BFHP, E and J are simply the translation generators $i\frac{\partial}{\partial\tau}$ and $i\frac{\partial}{\partial\psi}$, respectively. Therefore, the translations along the coordinates $\psi + \rho\tau$ and $\psi - \rho\tau$ are given by $J + \frac{1}{\rho}E$ and $J - \frac{1}{\rho}E$. Thus, taking the Penrose limit along the null geodesic $\psi+\rho\tau$ means taking the eigenvalues of $J+\frac{1}{\rho}E$ large, while keeping those of $J-\frac{1}{\rho}E$ fixed.

For $SU(2,2|4)$, we see that for large P, the eigenvalues of $(J+E)$ which are P-dependent become large, while those of $(J-E)$ which are P-independent stay finite. Thus, the parameter $\rho = 1$ is determined by the requirement that $J - \frac{1}{\rho}E$ be independent of P.

For the superalgebra $\mathfrak{osp}(8^*|4)$ with the even subsuperalgebra $\mathfrak{so}(6,2)\oplus\mathfrak{so}(5)$, the AdS energy operator E and the $U(1)_J$ generator J were given in equations (2.28) and (2.33):

$$E = \frac{1}{2}N_B + 2P\,, \qquad J = -\frac{1}{2}N_F + P\,. \tag{3.7}$$

It is clear that the P-independent linear combination of E and J is

$$J - \frac{1}{2}E = -\frac{1}{4}\left(N_B + 2N_F\right) \tag{3.8}$$

which corresponds to $\rho = 2$.

Similarly for the superalgebra $\mathfrak{osp}(8|4,\mathbb{R})$ with the even subsuperalgebra $\mathfrak{so}(8)\oplus\mathfrak{sp}(4,\mathbb{R})$, the generators E and J are (equations (2.42) and (2.46)):

$$E = \frac{1}{2}N_B + P + \epsilon\frac{1}{2}\,, \qquad J = -\frac{1}{2}N_F + 2P + \epsilon\,. \tag{3.9}$$

In this case, the P-independent linear combination can be taken as

$$J - 2E = -\frac{1}{2}\left(2N_B + N_F\right) \tag{3.10}$$

corresponding to $\rho = 1/2$.

We shall then define "renormalized" generators

$$\hat{g}^{(+1)} = \sqrt{\frac{\lambda}{P}}g^{(+1)}\,, \qquad \hat{g}^{(-1)} = \sqrt{\frac{\lambda}{P}}g^{(-1)} \tag{3.11}$$

belonging to the grade ±1 subspaces and take the limit $P \to \infty$ to obtain the PP-wave algebra (λ being a freely adjustable parameter).

It is evident that in this limit, the generators belonging to the subspace $\left(g^{(-1)} \oplus g^{(+1)}\right)$ become the generators of a super-Heisenberg algebra. The generators in $g^{(0)}$ subspace, that do not depend explicitly on P (assuming all the generators are in normal ordered form), will survive this limit intact.

Thus the PP-wave algebras obtained from the AdS/Conformal superalgebras are the semi-direct sums of a compact subsuperalgebra $\hat{\mathfrak{g}}^{(0)} = \mathfrak{g}^{(0)}/(J+\frac{1}{\rho}E)$ with a super-Heisenberg algebra $(\hat{\mathfrak{g}}^{(-1)} \oplus (J + \frac{1}{\rho}E) \oplus \hat{\mathfrak{g}}^{(+1)})$. The P-dependent U(1) generator $(J + \frac{1}{\rho}E)$ becomes the central charge in this limit.

4. Contraction of $\mathfrak{su}(2,2|4)$ and the PP-wave spectrum in the oscillator formalism

Consider now the realization of $SU(2,2|4)$ as bilinears of super-oscillators $\vec{\xi}_A$, $\vec{\eta}_M$ (and $\vec{\xi}^A$, $\vec{\eta}^M$) given in section (2.1.3). Define the generators of contracted algebra as

$$\hat{A}_{AM} = \lim_{P\to\infty} \sqrt{\frac{\lambda}{P}} \vec{\xi}_A \cdot \vec{\eta}_M \,, \qquad \hat{A}_{NB} = \lim_{P\to\infty} \sqrt{\frac{\lambda}{P}} \vec{\eta}^N \cdot \vec{\xi}^B \,. \tag{4.1}$$

They generate a super-Heisenberg algebra

$$\begin{aligned}\left[\hat{A}_{AM}, \hat{A}^{NB}\right] &= \lim_{P\to\infty} \frac{\lambda}{P} \left[\frac{1}{2}\delta_A^B \delta_M^N (J+E) + \right. \\ &\qquad\qquad \left. + \text{normal ordered } P \text{ independent generators in } \mathfrak{g}^{(0)}\right] \\ &= \lambda \delta_A^B \delta_M^N \,.\end{aligned} \tag{4.2}$$

In the PP-wave limit, the U(1) generator $\frac{1}{2}(J+E)$ becomes a central charge, labeled by the c-number λ. The $\mathfrak{g}^{(0)}$ subspace, modulo the generator $\frac{1}{2}(J+E)$, is the compact subsuperalgebra $\mathfrak{psu}(2|2) \oplus \mathfrak{psu}(2|2) \oplus \mathfrak{u}(1)$ and therefore the maximally supersymmetric PP-wave algebra obtained from $SU(2,2|4)$ is

$$[\mathfrak{psu}(2|2) \oplus \mathfrak{psu}(2|2) \oplus \mathfrak{u}(1)] \circledS \mathfrak{H}^{16,16}$$

where $\mathfrak{H}^{16,16}$ is the super-Heisenberg algebra with 16 even (bosonic) and 16 odd (fermionic) generators (plus the central charge λ) and $\circledS$ stands for semi-direct sum.

The P-independent U(1) factor belonging to the $\hat{\mathfrak{g}}^{(0)}$ subspace is simply

$$H = E - J = \frac{1}{2}\left(N_B + N_F\right) \tag{4.3}$$

which can be identified with the hamiltonian (modulo an overall scale factor). We shall denote the eigenvalue of the hamiltonian H on the lowest weight vectors as $\mathcal{E}_0$. To construct the UIRs of the resulting PP-wave algebra, we choose again a set of states $|\hat{\Omega}\rangle$ that transforms irreducibly under $PSU(2|2) \times PSU(2|2) \times U(1)$ and is annihilated by $\hat{\mathfrak{g}}^{(-1)}$ generators. Then by acting on $|\hat{\Omega}\rangle$ with $\hat{\mathfrak{g}}^{(+1)}$ generators repeatedly, we obtain a UIR of the PP-wave algebra.

There are infinitely many such irreducible representations $|\hat{\Omega}\rangle$. If we choose $|\hat{\Omega}\rangle$ to be the vacuum state $|0\rangle$ of all the oscillators (bosonic and fermionic), then the UIR of the PP-wave algebra consists of the super Fock space of the super-oscillators $\hat{A}^{MA}$. In fact, the vacuum state $|0\rangle$ is the only $\hat{\mathfrak{g}}^{(0)}$ invariant state (with zero $\mathrm{U}(1)_H$ charge).

We should note that, as in the case of $\mathrm{SU}(2,2|4)$, the vacuum $|0\rangle$ leads to short BPS multiplet of the PP-wave algebra. The range of energy eigenvalues $\mathcal{E}_0$ of the hamiltonian H on the generic BPS supermultiplet of $\mathrm{SU}(2,2|4)$ defined by $|\Omega\rangle = |0\rangle$ is $\Delta\mathcal{E}_0 = 8$ in our normalization. The energy range of the supermultiplet of PP-wave algebra defined by $|\hat{\Omega}\rangle = |0\rangle$ is also $\Delta\mathcal{E}_0 = 8$ in the same units.

Further, we should stress the following important point. Even though $\mathrm{SU}(2,2|4)$ admits doubleton supermultiplets ($P = 1$) with energy range $\Delta\mathcal{E}_0 = 2$, massless supermultiplets ($P = 2$) with $\Delta\mathcal{E}_0 = 4$ and massive BPS supermultiplets ($P = 3$) with $\Delta\mathcal{E}_0 = 6$, in the PP-wave limit we find that there are no analogs of supermultiplets with $\Delta\mathcal{E}_0 < 8$ since we take the limit $P \to \infty$. Note that for $P \geq 4$, the energy range of BPS multiplets corresponding to the Kaluza-Klein modes of IIB supergravity is always $\Delta\mathcal{E}_0 = 8$.

As mentioned before, for $\mathrm{SU}(2,2|4)$ while the choice $|\Omega\rangle = |0\rangle$ leads to short BPS multiplets, one can also obtain other shortened as well as generic long supermultiplets by choosing $|\Omega\rangle$ appropriately. Similarly, one can construct the analogs of such shortened and generic long supermultiplets of the PP-wave algebra with $\Delta\mathcal{E}_0 \geq 8$ by choosing different $|\hat{\Omega}\rangle$.

Since the entire Kaluza-Klein spectrum of 10-dimensional supergravity over $\mathrm{AdS}_5 \times \mathrm{S}^5$ fits into short unitary supermultiplets of $\mathrm{SU}(2,2|4)$ with lowest representation $|\Omega\rangle$ chosen as the vacuum (with zero central charge), the spectrum of the PP-wave algebra must be the unitary supermultiplet obtained from the lowest representation $|\hat{\Omega}\rangle = |0\rangle$. We give the corresponding unitary supermultiplet in table 4. It agrees with the zero mode spectrum given in Metsaev and Tseytlin [2]. In appendix A, we give the dictionary between our oscillators and those in [2].

$\mathrm{SU}(2)_{j_1} \times \mathrm{SU}(2)_{k_1} \times \mathrm{SU}(2)_{j_2} \times \mathrm{SU}(2)_{k_2}$ Young tableau	Eigenvalues of H $(= \mathcal{E}_0 + k)$	$\mathrm{SO}(4) = \mathrm{SU}(2)_{j_1} \times \mathrm{SU}(2)_{j_2}$ labels	$\mathrm{SO}(4)' = \mathrm{SU}(2)_{k_1} \times \mathrm{SU}(2)_{k_2}$ labels	Field of UIR of $\mathrm{U}(2,2|4)$	$\mathrm{U}(1)_Y$ quantum number
$\|0\rangle$	k	$(0,0)$	$(0,0)$	$\phi^{(1)}$	0
$\|\square,1,1,\square\rangle$	$k+1$	$(\frac{1}{2},0)$	$(0,\frac{1}{2})$	$\lambda_+^{(1)}$	$\frac{1}{2}$
$\|1,\square,\square,1\rangle$	$k+1$	$(0,\frac{1}{2})$	$(\frac{1}{2},0)$	$\lambda_-^{(1)}$	$-\frac{1}{2}$
$\|\square,\square,\square,\square\rangle$	$k+2$	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$	$A_\mu^{(1)}$	0
$\left\|\square\square,1,1,\begin{smallmatrix}\square\\\square\end{smallmatrix}\right\rangle$	$k+2$	$(1,0)$	$(0,0)$	$A_{\mu\nu}^{(1)}$	1
$\left\|1,\square\square,\begin{smallmatrix}\square\\\square\end{smallmatrix},1\right\rangle$	$k+2$	$(0,0)$	$(1,0)$	$\bar{\phi}^{(2)}$	-1

Table 4: Continued.

JHEP10(2002)007

$\left\vert 1, \yng(1,1), \yng(2), 1 \right\rangle$	$k+2$	$(0,1)$	$(0,0)$	$\tilde{A}^{(1)}_{\mu\nu}$	-1
$\left\vert \yng(1,1), 1, 1, \yng(2) \right\rangle$	$k+2$	$(0,0)$	$(0,1)$	$\phi^{(2)}$	1
$\left\vert \yng(2,1), 1, 1, \yng(2,1) \right\rangle$	$k+3$	$(\frac{1}{2},0)$	$(0,\frac{1}{2})$	$\lambda^{(2)}_{+}$	$\frac{3}{2}$
$\left\vert 1, \yng(2,1), \yng(2,1), 1 \right\rangle$	$k+3$	$(0,\frac{1}{2})$	$(\frac{1}{2},0)$	$\lambda^{(2)}_{-}$	$-\frac{3}{2}$
$\left\vert \yng(2), \yng(1), \yng(1), \yng(1,1) \right\rangle$	$k+3$	$(1,\frac{1}{2})$	$(\frac{1}{2},0)$	$\psi^{(1)}_{+\mu}$	$\frac{1}{2}$
$\left\vert \yng(1), \yng(1,1), \yng(2), \yng(1) \right\rangle$	$k+3$	$(\frac{1}{2},1)$	$(0,\frac{1}{2})$	$\psi^{(1)}_{-\mu}$	$-\frac{1}{2}$
$\left\vert \yng(1), \yng(2), \yng(1,1), \yng(1) \right\rangle$	$k+3$	$(\frac{1}{2},0)$	$(1,\frac{1}{2})$	$\lambda^{(3)}_{+}$	$-\frac{1}{2}$
$\left\vert \yng(1,1), \yng(1), \yng(1), \yng(2) \right\rangle$	$k+3$	$(0,\frac{1}{2})$	$(\frac{1}{2},1)$	$\lambda^{(3)}_{-}$	$\frac{1}{2}$
$\left\vert \yng(2,2), 1, 1, \yng(2,2) \right\rangle$	$k+4$	$(0,0)$	$(0,0)$	$\phi^{(3)}$	2
$\left\vert 1, \yng(2,2), \yng(2,2), 1 \right\rangle$	$k+4$	$(0,0)$	$(0,0)$	$\bar{\phi}^{(3)}$	-2
$\left\vert \yng(2), \yng(1,1), \yng(2), \yng(1,1) \right\rangle$	$k+4$	$(1,1)$	$(0,0)$	$h_{\mu\nu}$	0
$\left\vert \yng(2,1), \yng(1), \yng(1), \yng(2,1) \right\rangle$	$k+4$	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$	$A^{(2)}_{\mu}$	1
$\left\vert \yng(1), \yng(2,1), \yng(2,1), \yng(1) \right\rangle$	$k+4$	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$	$\bar{A}^{(2)}_{\mu}$	-1
$\left\vert \yng(2), \yng(2), \yng(1,1), \yng(1,1) \right\rangle$	$k+4$	$(1,0)$	$(1,0)$	$A^{(2)}_{\mu\nu}$	0
$\left\vert \yng(1,1), \yng(1,1), \yng(2), \yng(2) \right\rangle$	$k+4$	$(0,1)$	$(0,1)$	$\tilde{A}^{(2)}_{\mu\nu}$	0
$\left\vert \yng(1,1), \yng(2), \yng(1,1), \yng(2) \right\rangle$	$k+4$	$(0,0)$	$(1,1)$	$\phi^{(4)}$	0
$\left\vert \yng(1), \yng(2,2), \yng(2,2), \yng(1) \right\rangle$	$k+5$	$(\frac{1}{2},0)$	$(0,\frac{1}{2})$	$\lambda^{(4)}_{+}$	$-\frac{3}{2}$
$\left\vert \yng(2,2), \yng(1), \yng(1), \yng(2,2) \right\rangle$	$k+5$	$(0,\frac{1}{2})$	$(\frac{1}{2},0)$	$\lambda^{(4)}_{-}$	$\frac{3}{2}$
$\left\vert \yng(2), \yng(2,1), \yng(2,1), \yng(1,1) \right\rangle$	$k+5$	$(1,\frac{1}{2})$	$(\frac{1}{2},0)$	$\psi^{(2)}_{+\mu}$	$-\frac{1}{2}$
$\left\vert \yng(2,1), \yng(1,1), \yng(2), \yng(2,1) \right\rangle$	$k+5$	$(\frac{1}{2},1)$	$(0,\frac{1}{2})$	$\psi^{(2)}_{-\mu}$	$\frac{1}{2}$
$\left\vert \yng(2,1), \yng(2), \yng(1,1), \yng(2,1) \right\rangle$	$k+5$	$(\frac{1}{2},0)$	$(1,\frac{1}{2})$	$\lambda^{(5)}_{+}$	$\frac{1}{2}$
$\left\vert \yng(1,1), \yng(2,1), \yng(2,1), \yng(2) \right\rangle$	$k+5$	$(0,\frac{1}{2})$	$(\frac{1}{2},1)$	$\lambda^{(5)}_{-}$	$-\frac{1}{2}$
$\left\vert \yng(2), \yng(2,2), \yng(2,2), \yng(1,1) \right\rangle$	$k+6$	$(1,0)$	$(0,0)$	$A^{(3)}_{\mu\nu}$	-1
$\left\vert \yng(2,2), \yng(1,1), \yng(2), \yng(2,2) \right\rangle$	$k+6$	$(0,1)$	$(0,0)$	$\tilde{A}^{(3)}_{\mu\nu}$	1
$\left\vert \yng(1,1), \yng(2,2), \yng(2,2), \yng(2) \right\rangle$	$k+6$	$(0,0)$	$(0,1)$	$\phi^{(5)}$	-1
$\left\vert \yng(2,2), \yng(2), \yng(1,1), \yng(2,2) \right\rangle$	$k+6$	$(0,0)$	$(1,0)$	$\bar{\phi}^{(5)}$	1
$\left\vert \yng(2,1), \yng(2,1), \yng(2,1), \yng(2,1) \right\rangle$	$k+6$	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$	$A^{(3)}_{\mu}$	0

Table 4: Continued.

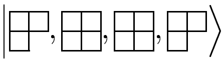
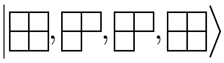
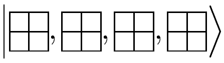
	$k+7$	$(\frac{1}{2},0)$	$(0,\frac{1}{2})$	$\lambda_{+}^{(6)}$	$-\frac{1}{2}$
	$k+7$	$(0,\frac{1}{2})$	$(\frac{1}{2},0)$	$\lambda_{-}^{(6)}$	$\frac{1}{2}$
	$k+8$	$(0,0)$	$(0,0)$	$\phi^{(6)}$	0

Table 4: Above we give the zero mode spectrum of the IIB superstring in the PP-wave limit. The entire zero mode spectrum corresponds to the positive energy supermultiplet of the PP-wave algebra obtained by choosing the vacuum state $|0\rangle$ of all bosonic and fermionic oscillators as the lowest weight vector. The first column is exactly as in table 1 with the states arranged in ascending eigenvalues of H ($k = 0, 1, 2, \ldots$). $\mathcal{E}_0$ is the eigenvalue of H on the states generated by the action of the supersymmetry generators in $\hat{\mathfrak{g}}^{(+1)}$ on the lowest weight vector $|0\rangle$. The higher eigenvalues of H ($k = 1, 2, \ldots$) correspond to energies of the excited states obtained by the action of bosonic generators in $\hat{\mathfrak{g}}^{(+1)}$ on these states.

5. Contraction of M-theory superalgebras $\mathfrak{osp}(8^*|4)$ and $\mathfrak{osp}(8|4,\mathbb{R})$ and the PP-wave spectrum in the oscillator formalism

Consider now the symmetry superalgebra $\mathfrak{osp}(8^*|4)$ of M-theory on $\mathrm{AdS}_7 \times \mathrm{S}^4$ as given in section 2.2. We obtain the corresponding PP-wave algebra by taking the limit $P \to \infty$. In this limit, we define the "renormalized" generators

$$\hat{A}_{AB} = \sqrt{\frac{\lambda}{2P}} A_{AB}\,, \qquad \hat{A}^{AB} = \sqrt{\frac{\lambda}{2P}} A^{AB}\,. \tag{5.1}$$

They close into the $\mathfrak{u}(1)$ generator $(J + \frac{1}{2}E)$, which becomes a central charge in the limit $P \to \infty$:

$$\begin{aligned}\left[\hat{A}_{AB}, \hat{A}^{CD}\right] &= \lim_{P\to\infty} \frac{\lambda}{P}\left[\frac{1}{2}\left(\delta_A^C\delta_B^D - (-1)^{(\deg B)(\deg D)}\delta_A^D\delta_B^C\right)\left(J + \frac{1}{2}E\right) + \right. \\ &\qquad\qquad \left. + \text{ normal ordered } P \text{ independent generators in } \mathfrak{g}^{(0)}\right] \\ &= \lambda\left(\delta_A^C\delta_B^D - (-1)^{(\deg B)(\deg D)}\delta_A^D\delta_B^C\right)\end{aligned} \tag{5.2}$$

and form a super-Heisenberg algebra $\mathfrak{H}^{18,16}$ with 18 bosonic and 16 fermionic generators (plus the central charge λ). The grade-(0) subspace $\hat{\mathfrak{g}}^{(0)} = \mathfrak{g}^{(0)}/(J + \frac{1}{2}E)$ form the subsuperalgebra $\mathfrak{su}(4|2)$. Hence the resulting PP-wave algebra is

$$\mathfrak{su}(4|2) \circledS \mathfrak{H}^{18,16}\,.$$

The $\mathfrak{u}(1)$ generator belonging to $\mathfrak{su}(4|2)$ is (see equation (3.8))

$$H = \frac{4}{3}\left(\frac{1}{2}E - J\right) = \frac{1}{3}\left(N_B + 2N_F\right) \tag{5.3}$$

which satisfies the following commutation relations:[4]

$$\left[H, \hat{A}_{ij}\right] = -\frac{2}{3}\hat{A}_{ij} \qquad \left[H, \hat{A}^{ij}\right] = \frac{2}{3}\hat{A}^{ij} \tag{5.4}$$

[4]Note that our generator H corresponds to $4h/\mu$ in BMN.

$$
\begin{aligned}
\left[H, \hat{A}_{\kappa\rho}\right] &= -\frac{4}{3}\hat{A}_{\kappa\rho} & \left[H, \hat{A}^{\kappa\rho}\right] &= \frac{4}{3}\hat{A}^{\kappa\rho} \\
\left[H, \hat{A}_{i\kappa}\right] &= -\hat{A}_{i\kappa} & \left[H, \hat{A}^{i\kappa}\right] &= \hat{A}^{i\kappa}\,.
\end{aligned} \tag{5.5}
$$

For the M-theory superalgebra $\mathfrak{osp}(8|4,\mathbb{R})$ on $\mathrm{AdS}_4 \times \mathrm{S}^7$, the corresponding PP-wave algebra has a similar form:

$$\mathfrak{su}(2|4) \circledS \mathfrak{H}^{18,16}$$

and the $\mathfrak{u}(1)$ generator in the $\hat{\mathfrak{g}}^{(0)}$ subspace $\mathfrak{su}(2|4)$ is

$$H = \frac{2}{3}(2E - J) = \frac{1}{3}(2N_B + N_F)\ . \tag{5.6}$$

It satisfies the following commutation relations:

$$
\begin{aligned}
\left[H, \hat{A}_{ij}\right] &= -\frac{4}{3}\hat{A}_{ij} & \left[H, \hat{A}^{ij}\right] &= \frac{4}{3}\hat{A}^{ij} \\
\left[H, \hat{A}_{\kappa\rho}\right] &= -\frac{2}{3}\hat{A}_{\kappa\rho} & \left[H, \hat{A}^{\kappa\rho}\right] &= \frac{2}{3}\hat{A}^{\kappa\rho} \\
\left[H, \hat{A}_{i\kappa}\right] &= -\hat{A}_{i\kappa} & \left[H, \hat{A}^{i\kappa}\right] &= \hat{A}^{i\kappa}\,.
\end{aligned} \tag{5.7}
$$

Thus the PP-wave limits of superalgebras $\mathfrak{osp}(8^*|4)$ and $\mathfrak{osp}(8|4,\mathbb{R})$ are isomorphic and agree with the PP-wave algebra given in BMN [8].

In table 5, we give the spectrum of the zero mode sector of the M-theory on PP-wave background by taking the vacuum state $|0\rangle$ of all the bosonic and fermionic oscillators as the lowest representation $|\hat{\Omega}\rangle$ of the PP-wave algebra. Again, $\mathcal{E}_0$ is the eigenvalue of the hamiltonian H on lowest weight vectors and the energy range is $\Delta\mathcal{E}_0 = 8$ for the PP-wave supermultiplet. There are no analogs of supermultiplets with $\Delta\mathcal{E}_0 < 8$.

Eigenvalues of H $(= \mathcal{E}_0 + \frac{2}{3}k)$	$SU(4) \times SU(2)$ Young tableau	$SU(4)$ Dynkin labels	$SU(2)$ spin
$\frac{2}{3}k$	$\lvert 0\rangle$	$(0,0,0)$	0
$\frac{2}{3}k+1$	$\lvert \yng(1),\yng(1)\rangle$	$(1,0,0)$	$\frac{1}{2}$
$\frac{2}{3}k+2$	$\lvert \yng(2),\yng(1,1)\rangle$	$(2,0,0)$	0
$\frac{2}{3}k+2$	$\lvert \yng(1,1),\yng(2)\rangle$	$(0,1,0)$	1
$\frac{2}{3}k+3$	$\lvert \yng(2,1),\yng(2,1)\rangle$	$(1,1,0)$	$\frac{1}{2}$
$\frac{2}{3}k+3$	$\lvert \yng(1,1,1),\yng(3)\rangle$	$(0,0,1)$	$\frac{3}{2}$
$\frac{2}{3}k+4$	$\lvert \yng(2,2),\yng(2,2)\rangle$	$(0,2,0)$	0
$\frac{2}{3}k+4$	$\lvert \yng(2,1,1),\yng(3,1)\rangle$	$(1,0,1)$	1

Table 5: Continued.

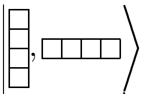
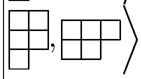
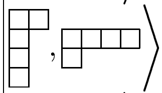
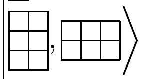
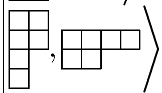
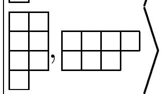
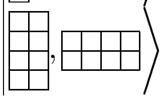
$\frac{2}{3}k+4$		$(0,0,0)$	2
$\frac{2}{3}k+5$		$(0,1,1)$	$\frac{1}{2}$
$\frac{2}{3}k+5$		$(1,0,0)$	$\frac{3}{2}$
$\frac{2}{3}k+6$		$(0,0,2)$	0
$\frac{2}{3}k+6$		$(0,1,0)$	1
$\frac{2}{3}k+7$		$(0,0,1)$	$\frac{1}{2}$
$\frac{2}{3}k+8$		$(0,0,0)$	0

Table 5: The spectrum of the zero mode sector of the PP-wave algebra of M-theory obtained by choosing the vacuum state $|0\rangle$ of all bosonic and fermionic oscillators as the lowest weight vector ($k = 0, 1, 2, \dots$). $\mathcal{E}_0$ is the eigenvalue of H on the states generated by the action of the supersymmetry generators in $\hat{\mathfrak{g}}^{(+1)}$ on the lowest weight vector $|0\rangle$. Note that the higher eigenvalues of H ($k = 1, 2, \dots$) correspond to energies of the excited states generated by the action of bosonic generators in the $\hat{\mathfrak{g}}^{(+1)}$ space on these states.

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A. Light-cone quantization of IIB superstring and the oscillator formalism

Here we give a dictionary between our generators of the PP-wave algebra of IIB superstring theory and those in the works of Metsaev and Tseytlin [2].

The world-sheet action for the type IIB superstrings propagating in the PP-wave background geometry was written in [1] using the coset construction. This action simplifies to free massive superstrings in the light-cone gauge, rendering it possible to find the spectrum. This was done in [2], where the spectrum was organized into the multiplets of $\mathrm{SO}(4) \times \mathrm{SO}(4)'$ with ever increasing energy eigenvalues.

To spell out conventions we briefly quote expressions relevant to this section from [2], assuming the relative strength of 5-form related to equation (1.1) is 1. The grade-(+1) space (with respect to the hamiltonian H) nilpotent subalgebra $\hat{\mathfrak{g}}^{(+1)}$ of the PP-wave Lie superalgebra is spanned by

$$\begin{aligned} P^I + J^{+I} &= \sqrt{2\lambda}a_0^I \qquad \text{for all} \quad I = 1,\ldots,8 \\ \mathcal{P}_L Q^+ &= \sqrt{\frac{\lambda}{2}}\left(\mathbb{I} - \bar{\Pi}\right)\bar{\gamma}^- \theta_0 \\ \mathcal{P}_R \bar{Q}^+ &= \sqrt{\frac{\lambda}{2}}\left(\mathbb{I} + \bar{\Pi}\right)\bar{\gamma}^- \bar{\theta}_0 \end{aligned} \tag{A.1}$$

and the $\hat{\mathfrak{g}}^{(-1)}$ nilpotent subalgebra by

$$\begin{aligned} P^I - J^{+I} &= \sqrt{2\lambda}\bar{a}_0^I \qquad \text{for all} \quad I = 1,\ldots,8 \\ \mathcal{P}_R Q^+ &= \sqrt{\frac{\lambda}{2}}\left(\mathbb{I} + \bar{\Pi}\right)\bar{\gamma}^- \theta_0 \\ \mathcal{P}_L \bar{Q}^+ &= \sqrt{\frac{\lambda}{2}}\left(\mathbb{I} - \bar{\Pi}\right)\bar{\gamma}^- \bar{\theta}_0 \,. \end{aligned} \tag{A.2}$$

Zero mode oscillators of the free massive strings a_0, $\bar{a}_0$, θ_0 and $\bar{\theta}_0$ satisfy the following (anti-)commutation relations:

$$\left[\bar{a}_0^I, a_0^J\right] = \delta^{IJ}\,, \qquad \left\{\bar{\theta}_0^\alpha, {\theta_0}^\beta\right\} = \frac{1}{4}\left(\gamma^+\right)^{\alpha\beta} \tag{A.3}$$

while all the other (anti-)commutators vanish. Spinors θ_0 and $\bar{\theta}_0$ have positive and negative chirality, respectively, and the 16-component Weyl spinors subject to fermionic light-cone gauge constraints $\gamma^+\theta_0 = 0$, $\gamma^+\bar{\theta}_0 = 0$. The commutators between the generators of the two subalgebras $\hat{\mathfrak{g}}^{(\pm 1)}$ produce:

$$\begin{aligned} \left[P^I - J^{+I}, P^J + J^{+J}\right] &= 2\lambda\delta^{IJ} \\ \left\{\mathcal{P}_L Q^+, \mathcal{P}_L \bar{Q}^+\right\} &= \frac{\lambda}{2}\left(\mathbb{I} - \bar{\Pi}\right)\bar{\gamma}^- \\ \left\{\mathcal{P}_R Q^+, \mathcal{P}_R \bar{Q}^+\right\} &= \frac{\lambda}{2}\left(\mathbb{I} + \bar{\Pi}\right)\bar{\gamma}^- \,. \end{aligned} \tag{A.4}$$

Here λ is the eigenvalue of the central charge $\frac{1}{2}(J+E)$. Subspaces $\hat{\mathfrak{g}}^{(\pm 1)}$ form a representation of $\hat{\mathfrak{g}}^{(0)} = \mathfrak{psu}(2|2) \oplus \mathfrak{psu}(2|2) \oplus \mathfrak{u}(1)$ subalgebra spanned by

$$\begin{aligned} -P^- &= H = a_0^I \bar{a}_0^I + \theta_L \bar{\gamma}^- \bar{\theta}_L + \bar{\theta}_R \bar{\gamma}^- \theta_R + \text{higher string mode contributions} \\ J^{IJ} &= a_0^I \bar{a}_0^J - a_0^J \bar{a}_0^I - \frac{i}{2}\bar{\theta}_L \bar{\gamma}^- \gamma^{IJ} \theta_L - \frac{i}{2}\theta_R \bar{\gamma}^- \gamma^{IJ} \bar{\theta}_R + \text{higher modes} \\ Q^- &= 2\left(a_0^I \bar{\gamma}^I \theta_R + \bar{a}_0^I \bar{\gamma}^I \theta_L\right) + \text{higher modes} \\ \bar{Q}^- &= 2\left(\bar{a}_0^I \bar{\gamma}^I \bar{\theta}_R + a_0^I \bar{\gamma}^I \bar{\theta}_L\right) + \text{higher modes,} \end{aligned} \tag{A.5}$$

where $\theta_{L/R} = \mathcal{P}_{L/R}\theta$ and J^{IJ} vanishes unless I and J both lie within either $1,\ldots,4$ or $5,\ldots,8$. The vacuum of the IIB superstring in the PP-wave background is a state annihilated by $\hat{\mathfrak{g}}^{(-1)}$, namely

$$\bar{a}_0^I\,|0\rangle = 0\,, \qquad \theta_L\,|0\rangle = 0\,, \qquad \bar{\theta}_R\,|0\rangle = 0 \tag{A.6}$$

and the Fock space is built by repeatedly acting on the PP-wave vacuum with the oscillators of $\hat{\mathfrak{g}}^{(+1)}$.

Thus our rescaled bilinears $\hat{A}_{AM}$ and $\hat{A}^{NB}$ (equation (4.1)) in the limit $P \to \infty$ correspond to the above zero mode oscillators of the IIB superstring in the PP-wave background. They are easy to write down explicitly. a_0^I and $\bar{a}_0^I$ can be realized as

$$\begin{aligned} a_0^I &= \begin{cases} \sqrt{\frac{1}{2P}}\left(\sigma^I\right)_{ir}\vec{a}^i\cdot\vec{b}^r & I=1,\dots,4 \\ \sqrt{\frac{1}{2P}}\left(\sigma^{I-4}\right)_{\gamma\mu}\vec{\alpha}^\gamma\cdot\vec{\beta}^\mu & I=5,\dots,8 \end{cases} \\ \bar{a}_0^I &= \begin{cases} \sqrt{\frac{1}{2P}}\left(\bar{\sigma}^I\right)^{ri}\vec{b}_r\cdot\vec{a}_i & I=1,\dots,4 \\ \sqrt{\frac{1}{2P}}\left(\bar{\sigma}^{I-4}\right)^{\mu\gamma}\vec{\beta}_\mu\cdot\vec{\alpha}_\gamma & I=5,\dots,8 \end{cases} . \end{aligned} \tag{A.7}$$

σ^I and $\bar{\sigma}^I$ above are given by

$$\sigma^I = (\mathbb{I}, i\vec{\sigma})\ , \qquad \bar{\sigma}^I = (\mathbb{I}, -i\vec{\sigma}) \tag{A.8}$$

where $\vec{\sigma}$ are the Pauli matrices. In this $P \to \infty$ limit, they satisfy

$$\left[\bar{a}_0^I, a_0^J\right] = \delta^{IJ}\,. \tag{A.9}$$

Similarly, the 16-component spinors θ_0^α are given by

$$\theta_0^\alpha = \begin{pmatrix} \psi_0^a \\ \mathbf{0} \end{pmatrix}, \qquad \bar{\theta}_0^\alpha = \begin{pmatrix} \bar{\psi}_0^a & \mathbf{0} \end{pmatrix} \tag{A.10}$$

where $a = 1, \dots, 8$ and

$$\begin{aligned} \psi_0^a &= \begin{cases} \sqrt{\frac{1}{8P\sqrt{2}}}\left(\left(\sigma^a\right)_{i\mu}\vec{a}^i\cdot\vec{\beta}^\mu + i\left(\sigma^a\right)_{\gamma r}\vec{\alpha}^\gamma\cdot\vec{b}^r\right) & a=1,\dots,4 \\ \sqrt{\frac{1}{8P\sqrt{2}}}\left(\left(\bar{\sigma}^{a-4}\right)^{\mu i}\vec{\beta}_\mu\cdot\vec{a}_i + i\left(\bar{\sigma}^{a-4}\right)^{r\gamma}\vec{b}_r\cdot\vec{\alpha}_\gamma\right) & a=5,\dots,8 \end{cases} \\ \bar{\psi}_0^a &= \begin{cases} \sqrt{\frac{1}{8P\sqrt{2}}}\left(\left(\bar{\sigma}^a\right)^{\mu i}\vec{\beta}_\mu\cdot\vec{a}_i - i\left(\bar{\sigma}^a\right)^{r\gamma}\vec{b}_r\cdot\vec{\alpha}_\gamma\right) & a=1,\dots,4 \\ \sqrt{\frac{1}{8P\sqrt{2}}}\left(\left(\sigma^{a-4}\right)_{i\mu}\vec{a}^i\cdot\vec{\beta}^\mu - i\left(\sigma^{a-4}\right)_{\gamma r}\vec{\alpha}^\gamma\cdot\vec{b}^r\right) & a=5,\dots,8 \end{cases} . \end{aligned} \tag{A.11}$$

In the $P \to \infty$ limit, they satisfy

$$\left\{\bar{\theta}_0^\alpha, \theta_0^\beta\right\} = \frac{1}{4}\left(\gamma^+\right)^{\alpha\beta}\,. \tag{A.12}$$

We used the chiral representation of Γ-matrix algebra in $d = 10$, in which $\Pi = \Gamma^1\Gamma^2\Gamma^3\Gamma^4$ and $\Pi' = \Gamma^5\Gamma^6\Gamma^7\Gamma^8$ are diagonal.

$$\begin{aligned} \Gamma^0 &= \epsilon\otimes\mathbb{I}\otimes\mathbb{I}\otimes\mathbb{I}\otimes\mathbb{I}\,, & \Gamma^5 &= \sigma_1\otimes\epsilon\otimes\sigma_3\otimes\mathbb{I}\otimes\epsilon\,, \\ \Gamma^1 &= \sigma_1\otimes\epsilon\otimes\epsilon\otimes\epsilon\otimes\epsilon\,, & \Gamma^6 &= \sigma_1\otimes\epsilon\otimes\mathbb{I}\otimes\epsilon\otimes\sigma_1\,, \\ \Gamma^2 &= \sigma_1\otimes\epsilon\otimes\sigma_1\otimes\mathbb{I}\otimes\epsilon\,, & \Gamma^7 &= \sigma_1\otimes\epsilon\otimes\mathbb{I}\otimes\epsilon\otimes\sigma_3\,, \\ \Gamma^3 &= \sigma_1\otimes\epsilon\otimes\epsilon\otimes\sigma_1\otimes\mathbb{I}\,, & \Gamma^8 &= \sigma_1\otimes\sigma_1\otimes\mathbb{I}\otimes\mathbb{I}\otimes\mathbb{I}\,, \\ \Gamma^4 &= \sigma_1\otimes\epsilon\otimes\epsilon\otimes\sigma_3\otimes\mathbb{I}\,, & \Gamma^9 &= \sigma_1\otimes\sigma_3\otimes\mathbb{I}\otimes\mathbb{I}\otimes\mathbb{I}\,. \end{aligned} \tag{A.13}$$

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